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The Laminar Boundary Layer in Slip Flow

-by-

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SUMMARY

This report discusses the effects of the existence of a small, but finite, molecular mean-free-path on the steady air flow in the laminar boundary layer. The boundary conditions at the exposed surface are modified by the existence of a slip velocity and temperature-jump between the air and the surface, and a theory of first-order approximation is developed to account for the consequent modification to the shear stress and heat flux to the surface. The correction is obtained as a quantity of the order of $(1/R_x^{1/2})$ compared with the uncorrected values of the shear stress and heat flux derived from the usual assumption of continuum flow (R_x being the local Reynolds number). The results are applicable to a wide range of external flow conditions, but are inapplicable to the internal conditions of the flow near the origin of the boundary layer.

In particular, it is found that, in the absence of an external pressure gradient, the correction to the local skin friction coefficient is zero; and if, as well, the surface temperature is uniform, then the heat transfer coefficient (k_H) is reduced by an amount $(M/3R_x)$.

In numerical work the results of this paper will be found significant if the local Reynolds number is between about 10^2 and 10^4 but they can be used as a cruder approximation down to about $R_x = 10$.

A summary of the relevant physical theory of non-uniform gases, and a comparison with other theoretical discussions of the problem of boundary layer slip flow, is included.

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LIST OF SYMBOLS

A	finite number
B_t	coefficient of s^t in expression for $(T_w - T_{th}^*)/T_a$
C	defined in equation (II.3) of Appendix II.
C_f	$\int_0^1 c_f ds$, the skin friction drag coefficient
C_p	$= (p_\delta - p_a)/\frac{1}{2}\rho_a u_a^2$, the pressure coefficient
δF	incremental force
$F(\sigma, t)$	defined in equation (III.6) of Appendix III, and related to $F(t)$ as shown in (III.8)
$F(X)$	function defined in equation (II.4) of Appendix II, and evaluated in figure 2
$G(x, y)$	value of an arbitrary property at the point (x, y)
$G[x, y, T(x)]$	value of an arbitrary property of the boundary layer at (x, y) if the air is slipping at the surface ($y=0$) where the temperature at a distance x from the nose is $T(x)$ just within the surface.
$G^*[x, y, T(x)]$	value of an arbitrary property of the boundary layer at (x, y) if there is no slip of the air at the surface ($y=0$) where the gas temperature at a distance x from the nose is $T(x)$.
K	maximum value of Mc_f
L	characteristic dimension of surface length
M	$= u_a/a_a$, Mach number of free-stream
N	number of molecules per unit volume
$N(\eta)$	$s^{\frac{1}{2}} \textcircled{\eta}$
Nu	$Q_s L/k_a (T_{th}^* - T_w)$, the Nusselt Number
P	$= \frac{1}{2}(c_f R_x^{\frac{1}{2}}) \left(\frac{u_a}{u_\delta}\right)^{3/2}$
Q	rate of heat flux
R	$= \rho_a u_a L/\mu_a$, surface Reynolds number
R_x	$= \rho_a u_a x/\mu_a$, local Reynolds number
δS	surface area element
S	Sutherland's constant, in $\mu \propto T^{3/2}/(T+S)$
T	gas static temperature
T_{th}	surface temperature for zero heat transfer (the thermometer temperature)
U	velocity characteristic of that of free stream

a	$= \sqrt{\gamma p / \rho}$, the speed of sound
c	'persistence factor' (the definition by equation (4.15) is assumed).
c_f	$= \tau_s / \frac{1}{2} \rho_a u_a^2$, the local skin friction coefficient
c_p	gas specific heat at constant pressure
c_v	volume
f	the 'momentum transfer coefficient'
k	gas thermal conductivity
k_H	$= Q_s / \frac{1}{2} \rho_a u_a^3$, the local heat transfer coefficient
m	molecular mass
n	distance measured normal to surface
p	gas static pressure
q	gas mean speed
r	defined in equation (4.25)
$r(\sigma) =$	$r(\sigma, 0)$, the temperature recovery factor
$r(\sigma, \eta)$	defined in equations (IV.14) and (IV.15) of Appendix IV
r_c	surface radius of curvature
s	$= x/L$
t	power of s in expansion of $(T_w - T_{th}^*)/T_a$
u_s	surface slip speed
u, v	velocity components relative to surface along (x, y) axes
u', v'	velocity components relative to plane of zero-slip, along (x', y') axes
\bar{v}	mean molecular speed
x, y	system of orthogonal coordinates measured along and perpendicular to surface, with origin at nose of surface
x', y'	system of orthogonal coordinates measured along and perpendicular to plane of no-slip
$\Delta c_f =$	$c_f - c_f^*$
$\Delta k_H =$	$k_H - k_H^*$
(N)	defined by equation (IV.16) of Appendix IV
Φ	defined in equation (4.32)
α	the accommodation coefficient
β	a finite number

γ	=	c_p/c_v , the ratio of the gas specific heats
δ		length characteristic of boundary layer thickness
δ_1		the boundary layer displacement thickness
s		value of s where c_f^* exceeds the maximum value of c_f
ζ		the 'coefficient of slip' (see equation (I.7) of Appendix I)
η	=	y/δ , and defined more precisely in equation (IV.8) of Appendix IV
θ	=	T/T_a
λ		the gas mean free path at the surface
μ		the gas coefficient of viscosity
ξ	=	$(x-\xi_0)/L$
ξ_0, ξ'_0		arbitrarily chosen values of x and x' , respectively
ρ		the gas density ($= Nm$)
σ	=	$\mu c_p/k$, the Prandtl Number
τ		shear stress
ϕ'	=	$2(u/u_a) \left(= \frac{d}{d\eta} \phi(\eta) \right)$
ψ		non-dimensional stream function defined in equation (IV.3) of Appendix IV
ω	=	$\frac{T}{\mu} \frac{d\mu}{dT}$, assumed a constant

Subscript: 'a' refers to conditions in gas in the free stream
's' refers to conditions at the surface in the gas
'w' refers to conditions on the surface
' ζ ' refers to conditions at the plane of no-slip
($y = -\zeta(x)$)
' δ ' refers to conditions in gas outside boundary layer.

An asterisk denotes the value of a property derived on the assumption of continuum-flow (i.e. assuming $\lambda = 0$).

A prime is used to denote differentiation of a function with respect to its argument.

1. Introduction

The first outward manifestation of 'high-altitude effects' upon the structure of the boundary layer over a body is generally to be found in the occurrence of a certain velocity of slip of the air at the surface of the body. It is an established fact that a viscous gas tends to adhere to any moving surface with which it is in contact: if the gas is regarded as a continuum then the assumption of 'zero slip' can be accepted. It is this tendency to drag the air along with it which causes near a moving surface the appearance of a boundary layer of air, and within the boundary layer the viscous stresses are important.

If the gas is regarded, not as a homogeneous medium (i.e. as a continuum), but as an aggregation of molecules, this concept of 'zero-slip' breaks down, since the gas cannot be said to be continuously in contact with the surface. Likewise, the temperature of the gas at the wall is not necessarily that of the wall itself - there is a 'temperature jump'. The properties of individual molecules striking the surface are their properties derived from their previous collision, within the main bulk of the gas. In simplified terms, we might conceive that the molecules possess the same properties within a distance of one mean free path from the surface. The implications of this concept are generally unimportant for gases at ordinary temperature and pressures since the free path is very short. However, in the rarified air at high altitudes of flight, the occurrence of a slip velocity at the surface cannot be overlooked, as the mean free path may be appreciable compared with the boundary layer thickness.

Even at low altitudes, it might be expected that the effects would not be negligible near the origin of the boundary layer, where the thickness of the layer becomes very small.

This discussion concerns the correction to be applied to the estimated steady rate of heat transfer and skin friction to a two-dimensional surface, on the assumption that the mean free path is small compared with the dimensions of the boundary layer, but not negligible.

We shall first briefly state the physical laws governing the value of the slip velocity and the temperature jump at a surface - a more general discussion of the physics being given in the Appendix, for the benefit of those who may be unaware of the approximations involved; and then we shall attempt to find how far these laws - which were derived for simple shearing motion and one-dimensional convection of heat in a fluid at rest - may

be applicable to the flow in the boundary layer. After some modification, these laws are shown to give a new set of boundary conditions at the surface within the boundary layer, valid if the mean free path is small, so that all we have is a first approximation. It should be noted that if the mean free path is not small but commensurate with the boundary layer thickness, not only do these results break down, but also other assumptions inherent in the formation of the boundary layer equations become invalid.

We shall then show how these modifications to the boundary conditions affect the solution of the boundary layer equations. In doing so we need only assume that the flow is laminar near the surface, and we do not need to construct the solution of the equations; instead we relate it to other known solutions. These general results are then applied to some particular solutions for the laminar boundary layer, in a way which enables us to make certain quantitative deductions. A completely general set of results cannot be obtained, because of the difficulty of accounting precisely for the effect of the surface temperature jump. However, the results given cover a wide range of problems, and a comparison with other theoretical results is included.

2. The Results of the Kinetic Theory of Gases

As is shown in the Appendix, based on the theories expanded in refs. 1 and 2, in a one-dimensional shearing motion with a gas velocity q parallel to a fixed wall, physical theory leads one to expect that at the wall the molecules will possess a certain 'slip velocity'

$$u_s = 2c \left(\frac{2-f}{f} \right) \frac{dq}{dn} \lambda \quad \dots\dots\dots (2.1)$$

relative to the wall, and are not (as is usually assumed) brought to rest thereon. In this expression f is a factor called the 'momentum transfer coefficient' which must be found from experiment, dq/dn is the velocity gradient in the gas normal to the wall (measured with respect to time), and λ is the mean free path of the molecules. The factor c is sometimes called the 'persistence factor' and in fact $(c\lambda)$ may be related to the usual definition of viscosity

$$\mu = c \rho \bar{v} \lambda \quad \dots\dots\dots (2.2)$$

/where ...

where ρ is the gas density, and \bar{v} is the mean molecular speed.

Similarly if heat energy is being transported one-dimensionally through a gas at rest to a wall, there will be a difference in the mean temperature of the molecules at the surface (T_s) and that of the wall itself (T_w): this is a 'temperature jump' which is usually neglected and is found to be equal to

$$T_s - T_w = \frac{4}{\gamma+1} \frac{\sigma}{\sigma} \left(\frac{2-\alpha}{\alpha} \right) \frac{dT}{dn} \lambda \quad \dots\dots\dots (2.3)$$

where α is called the 'accommodation coefficient', σ is the Prandtl number ($c_p \mu / k$), and γ is the ratio of the gas specific heats (c_p / c_v). The temperature gradient (dT/dn) normal to the wall is assumed to have a constant mean value within the gas, as - in the statement of (2.1) - was the velocity gradient. In fact, it is only necessary that these restrictions apply within distances of the order of λ from the surface. At such distances from a surface of a boundary layer very similar conditions exist, as will now be shown.

3. Application to the Flow within the Boundary Layer

Taking axes normal to, and parallel with, a plane wall, the two-dimensional equations which govern the motion of a compressible boundary layer on this wall are: that of continuity:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad \dots\dots\dots (3.1)$$

those of conservation of momentum:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad \dots\dots\dots (3.2)$$

$$0 = \frac{\partial p}{\partial y} \quad \dots\dots\dots (3.3)$$

and finally that of conservation of energy, which may be written as

$$c_p \rho \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 + u \frac{dp}{dx} \quad \dots\dots\dots (3.4)$$

The symbols used here will be found defined in the list at the beginning of this report.

/These ...

These equations are obtained using the usual approximation that quantities of the order of (δ^2/L^2) are negligible in comparison with unity - where δ and L are quantities representative of the boundary layer thickness and surface length respectively; and that the non-linear terms in viscosity and conductivity may be ignored - an assumption which involves the neglect of terms of the order $(\mu U/pL)$ compared with unity. As is well-known, both these assumptions are justifiable if R , the Reynolds number, is large, and imply the neglect of terms of the order $(1/R)$ or (M^2/R) compared with unity. Here we shall regard M as a quantity of order unity[‡], so that in either case it is necessary only to regard R as large compared with unity to justify the assumptions: the modifications otherwise imposed, if M is very small or very large, are discussed in refs. 3 and 4, but are unimportant to the principles we wish to establish here.

Again, if the wall is curved and we measure x and y along and perpendicular to the curved surface, then the equations (3.1) - (3.4) still apply if we also neglect terms of the order of (δ/\bar{r}_c) compared with unity, where \bar{r}_c indicates the mean radius of curvature of the surface.

In the light of these approximations it is therefore pertinent to enquire if, say, the occurrence of a velocity of slip at the surface has any significance within the accuracy of the usual approximations. We have seen, in fact, from (2.1) that u_s is a quantity of the order of $\lambda \times$ velocity gradient normal to the surface: in the boundary layer at the surface, the normal velocity gradient is in fact a quantity of the order of u_a/δ , the suffix 'a' denoting free stream conditions. Hence

$$\frac{u_s}{u_a} = O\left(\frac{\lambda}{\delta}\right) \dots\dots\dots (3.5)$$

From (2.2), since \bar{v} is commensurate with the speed of sound,

$$\lambda = O\left(\frac{\mu}{\rho a}\right) = O\left(\frac{L}{R}\right) \dots\dots\dots (3.6)$$

whereas, as already noted

$$\delta = O\left(\frac{L}{R^{\frac{1}{2}}}\right) \dots\dots\dots (3.7)$$

/so that ...

‡ The statement that $M = O(1)$ here asserted does not of course preclude the condition that $M \ll 1$.

so that in (3.5)

$$\frac{u_s}{u_a} = O\left(\frac{1}{R^{\frac{1}{2}}}\right) \dots\dots\dots (3.8)$$

In general, the value of u derived from the equations of the boundary layer has an inaccuracy of the order of (u_a/R) , so that it follows that, from (3.6), u_s is certainly significantly different from zero within the accuracy of the approximations. Accordingly, we may use the boundary condition $u = u_s$ at the surface, instead of the usual condition of no-slip, without otherwise correcting the boundary layer equations: similarly we may include the effect of the temperature jump. However, plainly it is impossible to do more than consider the first-order effects of these new boundary conditions, as second-order effects would not increase the accuracy of the solution simply because there are other comparable effects which are neglected as well.

To establish whether the results of the physical theory (given in para. 2) are relevant to conditions near the surface in the boundary layer, we note immediately that the variation of, for instance, the velocity gradient in the boundary layer has an insignificant effect upon the value of the slip velocity, since such variations in the flow will introduce terms of correction to the expression (2.1) depending on (λ/δ) : i.e. the proportionate variation in the properties within distances of the order of the mean free path. From (3.5) and (3.8), the value of (u_s/u_a) is evidently a small quantity of the magnitude $1/R^{\frac{1}{2}}$, and so such corrections will only affect the estimation of its value by magnitudes of the order of $(1/R)$: within the accuracy of the boundary layer assumptions, such magnitudes are negligible. Hence, we may show that (2.1) is applicable to the conditions within the boundary layer if we put:

$$u_s = 2c \left(\frac{2-f}{f}\right) \left(\frac{\partial u}{\partial y}\right)_s \lambda \dots\dots\dots (3.9)$$

In the formulation of (2.3) variations in, for example, the temperature gradient, have been neglected, but this is likewise justifiable within the limits of our approximations: however, no account has so far been taken of the mean kinetic energy of the gas relative to the wall. The kinetic energy of the gas near the wall is a quantity of the order $\frac{1}{2} u_s^2$ per unit mass, whereas its excess heat energy is of the order $c_p (T_s - T_w)$: the ratio of these two quantities is, from (2.1) and (2.3)

$$\frac{1}{2} u_s^2 / c_p (T_s - T_w) = O \left[\left(\frac{dg}{dn} \right)^2 \lambda^2 / c_p \left(\frac{dT}{dn} \right) \right]$$

/The value ...

The value of the velocity gradient is a magnitude of the order (u_a/δ) , and the value of the temperature gradient is in general of order (T_a/δ) ; so that the mean kinetic energy is a small fraction of the difference between the heat energy of the incident molecules and the wall (since $u_a^2/c_p T_a = (\gamma-1)M^2$ is in general of order unity), the fraction being a quantity of the magnitude (λ/δ) . Similarly the rate at which work is done on the wall by the shearing stress is small compared with the rate of heat flux i.e.

$$\left[u_s \mu_s \left(\frac{\partial u}{\partial y} \right)_s / k_s \left(\frac{\partial T}{\partial y} \right)_s \right] = 0 \left(\frac{u_s}{u_a} \right) = 0 \left(\frac{1}{R^{\frac{1}{2}}} \right)$$

Thus the consideration of mean kinetic energy as well as of heat energy will not affect the formulation of the expression for the temperature jump, except in second order terms, which are negligible to the order of approximation used here; so that we may write (2.3) as the condition governing the temperature jump at the surface in the boundary layer:

$$T_s - T_w = \frac{4\gamma}{\gamma+1} \frac{c}{\sigma} \left(\frac{2-a}{a} \right) \left(\frac{\partial T}{\partial y} \right)_s \lambda \dots\dots\dots (3.10)$$

Now suppose that we may express the variation of any property (G) within the boundary layer near a point $x = \xi_0$ on the surface ($y=0$) by a series:

$$G(x,y) = G(\xi_0,0) + (x-\xi_0)G_x(\xi_0,0) + yG_y(\xi_0,0) + \dots$$

Put $(x-\xi_0)/L = \xi$ and $y/\delta = \eta$: then derivatives with respect to ξ and η are, in general, commensurate if (as is assumed) the quantity $\delta/L = O(1/R^{\frac{1}{2}})$ is small. Further let us consider the variation of the properties within distances of the order of a mean free path from the point of the surface $(\xi_0,0)$: suppose that $x-\xi_0 = A\lambda$, $y = \beta\lambda$, where A and β are finite. Then

$$G(x,y) = G(\xi_0,0) + A \left(\frac{\lambda}{L} \right) G_\xi(\xi_0,0) + \beta \left(\frac{\lambda}{\delta} \right) G_\eta(\xi_0,0) + \dots$$

Thus within the accuracy of the solution, neglecting terms of order $(1/R)$ compared with those of order unity,

$$G(\xi_0 + A\lambda, \beta\lambda) = G(\xi_0,0) + \beta \left(\frac{\lambda}{\delta} \right) G_\eta(\xi_0,0) \dots\dots\dots (3.11)$$

For example,

$$u(\xi_0 + A\lambda, \beta\lambda) = u_s(\xi_0) + \beta\lambda \cdot \left(\frac{\partial u}{\partial y} \right)_s \Big|_{x=\xi_0} \dots\dots\dots (3.12)$$

/and from ...

and from (3.9) in (3.12)

$$u(\xi_0 + A\lambda, \beta\lambda) = \left[2c \left(\frac{2-f}{f} \right) + \beta \right] \lambda \left(\frac{\partial u}{\partial y} \right)_s \Big|_{x=\xi_0} \dots\dots (3.13)$$

It follows that $u=0$ if

$$y = \beta\lambda = -2c \left(\frac{2-f}{f} \right) \lambda = -\zeta, \text{ say} \dots\dots\dots (3.14)$$

The length ζ , often termed the 'coefficient of slip', may therefore be envisaged as the distance below the surface where the mean velocity of the gas would be reduced to rest relative to the true surface. This apparent depression of the plane of 'zero slip' will in fact vary with $s = x/L$, as the mean free path will be related to the local gas temperature and the local wall temperature. However this variation is small, and in fact negligible over distances along the surface of the order λ , as demonstrated by (3.13).

At this plane of 'zero slip', the gas temperature is, from (3.11),

$$T_\zeta \Big|_{x=\xi_0} = T(\xi_0 + A\lambda, -\zeta) = T_s(\xi_0) - \zeta \left(\frac{\partial T}{\partial y} \right)_s \Big|_{x=\xi_0}$$

or, using (3.10) and (3.14),

$$(T_\zeta - T_w) \Big|_{x=\xi_0} = 2c \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-c}{\alpha\sigma} \right) - \left(\frac{2-f}{f} \right) \right] \left[\lambda \left(\frac{\partial T}{\partial y} \right)_s \right]_{x=\xi_0} \dots\dots (3.15)$$

and evidently at the plane of zero slip, the gas temperature is different from the wall temperature.

Again at the plane of zero slip, the normal mass flow is

$$(\rho v)_\zeta \Big|_{x=\xi_0} = (\rho v)_s \Big|_{x=\xi_0} + O \left(\frac{\rho_s u_s \zeta}{L} \right)$$

since, by (3.1), $\frac{\partial(\rho v)}{\partial y}$ at $y=0$ is equal to $\frac{-d(\rho_s u_s)}{dx}$.

But there can be no flow into the surface at $y=0$, and so from (3.5) and (3.6),

$$\frac{(\rho v)_\zeta}{\rho_a u_a} = O \left(\frac{\rho_s u_s \zeta}{\rho_a u_a L} \right) = O \left(\frac{\lambda}{bL} \right) = O \left(\frac{1}{R^{3/2}} \right) \dots\dots\dots (3.16)$$

Now (v/u_a) is in general of order $1/R^{1/2}$ in the boundary layer and so its value at the plane of 'no-slip' is zero, within the

/assumed ...

assumed accuracy. Hence, at the plane of 'no-slip' the gas is sensibly at rest relative to the surface.

The concept of a plane of 'zero slip' has of course no physical significance, as the flow does not exist below the true surface; however it is justifiable as a mathematical notion, based on an analytic continuation of the variation of the mean gas velocity deduced from the solution of the equations, and it is a notion which is a convenient one as it enables us to suggest a method of solution of the boundary layer equations in the condition of slip-flow. This method will be explained in the next paragraph.

Before embarking on these arguments, one fundamental point about this method of solution needs clarification. In equation (3.11), the property G is arbitrary: it might be the velocity u , say, or the shear stress, which involves $\frac{\partial u}{\partial y}$. Yet, if it refers to the velocity u , then as in (3.12) we assume that within distances of the order λ from the surface the velocity varies linearly with distance: this implies that (with constant viscosity) the shear stress is constant over this distance. It is apparently, therefore, incompatible to proceed as we shall do, to calculate the change in shear stress by the use of equation (3.11). However, there is no incompatibility. To illustrate this, let us consider a simple example.[‡] As a first approximation, equation (3.11) tells us that the shear stress varies linearly with distance close to the surface: assuming say that the viscosity is constant, then this admittedly implies a quadratic distribution of velocity. We have, however, that the shear stress at the true surface at a distance ζ from the surface of no-slip is

$$\tau_s = \tau_\zeta + \mu \zeta \left(\frac{\partial^2 u}{\partial y^2} \right)_\zeta$$

Now, $\mu \left(\frac{\partial^2 u}{\partial y^2} \right)$ at the surface of no-slip is equal to the pressure gradient along the surface $\frac{dp}{dx}$ (from the equation of motion of the boundary layer). Thus the change in the shear stress between the true surface and the apparent no-slip surface is

$$\tau_s - \tau_\zeta = \zeta \frac{dp}{dx}$$

which is a result we shall later derive again. To evaluate this difference we need to know ζ : a first approximation to ζ may be obtained, as in equation (3.14) by assuming a linear variation in velocity; thus even if we make the more acceptable assumption

/of a ...

[‡] The author is indebted to Mr. G.M. Lilley for suggesting this.

of a quadratic variation in velocity, we find that:

$$u_s = u_\zeta + \zeta \left(\frac{\partial u}{\partial y} \right)_\zeta + \frac{1}{2} \zeta^2 \left(\frac{\partial^2 u}{\partial y^2} \right)_\zeta$$

i.e. from (3.9), since $u_\zeta = 0$, $(\partial u / \partial y)_\zeta$, and $(\partial^2 u / \partial y^2)_\zeta$ are constant

$$2c \left(\frac{2-f}{f} \right) \left(\frac{\partial u}{\partial y} \right)_s \lambda = 2c \left(\frac{2-f}{f} \right) \left[\left(\frac{\partial u}{\partial y} \right)_\zeta + \zeta \left(\frac{\partial^2 u}{\partial y^2} \right)_\zeta \right] \lambda = \zeta \left[\left(\frac{\partial u}{\partial y} \right)_\zeta + \frac{1}{2} \zeta \left(\frac{\partial^2 u}{\partial y^2} \right)_\zeta \right]$$

$$\text{i.e. } \zeta = 2c \left(\frac{2-f}{f} \right) \lambda \left[1 + c \left(\frac{2-f}{f} \right) \lambda \left(\frac{1}{\tau_\zeta} \frac{dp}{dx} \right) + \dots \right]$$

The first-order approximation to ζ is adequate, and there is therefore no incompatibility in making the assumption of a linear variation in velocity which leads to this approximation. This assumption is sufficient to yield ζ .

4. The solution of the Boundary Layer Equations in Slip Flow

4.1. General Conclusions

Suppose we now introduce a system of orthogonal coordinates (x', y') , with the lines $y' = \text{constant}$ lying parallel to the plane of 'no-slip' ($y = -\zeta(x)$), and $y' = 0$ corresponding to the plane of 'no-slip'. The curvature of the lines $y' = \text{const.}$ is then simply proportional to $(d^2 \zeta / dx^2)$ since ζ is a small length, of the order λ : their radius of curvature is thus a magnitude

$$r_c = \frac{L^2}{\lambda} \quad O(1)$$

so that δ / r_c is of the order $(\delta \lambda / L^2)$ i.e. of the order $1/R^{3/2}$. Hence, in the light of the remarks which we made at the beginning of the last section, it follows that if we neglect terms of order $(1/R)$ compared with unity, we may write down the equations governing the flow within the boundary layer as

$$\frac{\partial(\rho u')}{\partial x'} + \frac{\partial(\rho v')}{\partial y'} = 0 \quad \dots\dots\dots (4.1)$$

$$\rho u' \frac{\partial u'}{\partial x'} + \rho v' \frac{\partial v'}{\partial y'} = - \frac{dp}{dx'} + \frac{\partial}{\partial y'} \left(\mu \frac{\partial u'}{\partial y'} \right) \quad \dots\dots\dots (4.2)$$

$$\frac{\partial p}{\partial y'} = 0 \quad \dots\dots\dots (4.3)$$

$$c_p \rho \left(u' \frac{\partial T}{\partial x'} + v' \frac{\partial T}{\partial y'} \right) = u' \frac{dp}{dx'} + \frac{\partial}{\partial y'} \left(k \frac{\partial T}{\partial y'} \right) + \mu \left(\frac{\partial u'}{\partial y'} \right)^2$$

..... (4.4)

where u' and v' are the velocity components parallel to the x' and y' axes. Except for the substitution of primed symbols, these are identical with equations (3.1) - (3.4).

Now, if - as is usual - the equations (3.1) - (3.4) are solved subject to the condition of no-slip at the true surface, the boundary conditions to be imposed are:

$$\begin{aligned} u = v = 0 \quad \text{and} \quad T = T_w \quad \text{at} \quad y = 0 \\ u = u_\infty \quad \text{and} \quad T = T_\infty \quad \text{at} \quad y = \infty \end{aligned}$$

..... (4.5)

the boundary conditions and the solution being valid only for $x > 0$ where $x=0$ corresponds with the point of origin of the boundary layer. Yet, if we allow the existence of a slip-velocity and introduce the new (x', y') axes, these have been chosen so that the corresponding boundary conditions are:

$$u' = v' = 0, \quad T = T_\infty, \quad \text{at} \quad y' = 0 \quad \text{..... (4.6a)}$$

since from (3.16) we found that the gas is at rest relative to the surface at the plane of no-slip. At $y = \infty$, corresponding to the condition $u = u_\infty$, we have that

$$u_\infty = \left(u' + v' \frac{dz}{dx} \right) / \sqrt{1 + \left(\frac{dz}{dx} \right)^2}$$

But $(dz/dx) = O(\lambda/L) = O(1/R)$, from (3.6): so that within the approximations of the solution, $u' = u_\infty$ at $y = \infty$, which is also the condition at $y' = \infty$. Hence we also have that

$$u' = u_\infty, \quad T = T_\infty \quad \text{at} \quad y' = \infty \quad \text{..... (4.6b)}$$

Without loss in generality, we may assert that these conditions are valid for $x' > 0$.

Comparing the equations (4.6) with those of (4.5), it follows that the boundary conditions, as well as the actual equations, are similar; whence we derive the following important result.

To the order of approximation implicit in the assumption of the usual boundary layer equations, in slip-flow the conditions at a point (x', y') , - referred to axes parallel and perpendicular to the plane of no-slip ($y' = 0$, or $y = -z(x)$), - are identical with

/those in ...

those in the boundary layer over the same surface at a point (x, y) , - referred to axes parallel and perpendicular to the surface $(y=0)$ where a condition of no-slip applies; - except that, in the former problem, the temperature of the gas at the plane of no-slip is different from that in the latter problem, where the temperature of the gas at the surface is that of the wall (i.e. there is no temperature jump).

Let us attempt to formulate this condition mathematically. Suppose we denote by $G^* [x, y, T(x)]$ a property at the point (x, y) in a flow without slip referred to axes parallel and perpendicular to the surface $(y=0)$ where the gas temperature is $T(x)$, and $x=0$ is the origin of the plate; and by $G [x, y, T(x)]$ the same property at the point (x, y) if the air is slipping at the surface, where the wall temperature is $T(x)$.

Now, in our notation, a point on the plane of no-slip $y'=0$, $x'=\xi'_0$, corresponds to the point $y = -\zeta(\xi_0)$, and $x=\xi_0$, where

$$\xi'_0 = \int_0^{\xi_0} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \xi_0 \left[1 + O\left(\frac{\zeta^2}{L^2}\right) \right]$$

i.e. where $\xi'_0 = \xi_0$, to the appropriate order of approximation. Hence, the conclusion already stated implies that

$$G [x, y, T_w(x)] = G^* [x, y + \zeta(x), T_\zeta(x)] \dots\dots\dots (4.7)$$

In particular, the value of properties on the surface where $y=0$, are given by

$$\begin{aligned} G [x, 0, T_w(x)] &= G^* [x, \zeta(x), T_\zeta(x)] \\ &= G^* [x, 0, T_\zeta(x)] + \zeta(x) \cdot G_y^* [x, 0, T_w(x)] \\ &\dots\dots\dots (4.8) \end{aligned}$$

using the results of (3.14). It will be noticed that in the small correction term multiplied with $\zeta(x)$ it is immaterial, to the order of approximation valid, whether the quantity G_y^* is that derived for a gas surface temperature of T_w or T_ζ , since the difference is small.

However, it is the difference between these two quantities which significantly affects the value of the first term on the righthand side of (4.8). We require to find the difference

$$\Delta G = G [x, 0, T_w(x)] - G^* [x, 0, T_w(x)]$$

/and to ...

and to do so, we evidently need to find the difference between $G^* [x, 0, T_w]$ and $G^* [x, 0, T_\zeta]$. We may require a knowledge of the former quantity - since this is the value of the property derived for a continuum flow with zero surface slip and temperature jump; but this is of no value in finding the latter, since $(T_w - T_\zeta)$ varies with x .

This may be seen from equation (3.15), which gives the value of $(T_\zeta - T_w)$. Only in the condition of zero heat-transfer, $(\frac{\partial T}{\partial y})_s = 0$, is the effect of this change in temperature easily accounted for, since then the temperature jump is zero. For other conditions, it is not easy exactly to calculate the difference between $G^* [x, 0, T_\zeta]$ and $G^* [x, 0, T_w]$. Take, for example, the case of a flat plate with a uniform surface temperature, $T_w = \text{const.}$: then λ does not change with x , but $(\frac{\partial T}{\partial y})_s$, and so also $(T_\zeta - T_w)$, varies in general as $1/\sqrt{x}$, and there is no general solution for the flow over a flat plate with a temperature distribution $T_\zeta = T_w + Bx^{-1/2}$ (where B is a constant) except at incompressible speeds of flow.

Fortunately, the conditions of zero heat transfer and low Mach number, are both important ones; and the effect of the temperature jump may otherwise be assessed in the particular condition in which the shear stress is independent of the temperature distribution: this is so in a compressible flow without a pressure gradient if the viscosity varies in proportion to the absolute temperature. Although this latter is a condition not exactly satisfied it does give an adequate approximation to the rate of heat transfer, and presumably also to the effect of the temperature-jump, over a flat plate, provided the Mach number is not too high.

4.2. The Solution for Zero Heat-Transfer

If there is no heat transfer to the surface

$$\left(\frac{\partial T}{\partial y}\right)_s = 0 \quad \dots\dots\dots (4.9)$$

and so from (3.15):

$$T_\zeta = T_w \quad \dots\dots\dots (4.10)$$

/Hence, ...

Hence, equation (4.8) may be written as

$$G [x, 0, T_w(x)] = G^* [x, 0, T_w(x)] + \zeta(x) G_y^* [x, 0, T_w(x)] \dots (4.11)$$

Consider now the shear stress in slip flow:

$$\tau = \left(\mu \frac{\partial u}{\partial y} \right)$$

Putting $G \equiv \tau$, we may in (4.11) immediately relate τ_s to τ_s^* (i.e. the shear stress at the wall, calculated for the same conditions with no slip at the surface.) In fact:

$$\tau_s = \tau_s^* + \zeta(x) \left(\frac{\partial \tau^*}{\partial y} \right)_s \dots (4.12)$$

But from (3.2), if the condition of no-slip is imposed, at the wall

$$\left(\frac{\partial \tau^*}{\partial y} \right)_s = \frac{dp}{dx} \dots (4.13)$$

so that

$$\Delta \tau_s = \tau_s - \tau_s^* = \zeta \frac{dp}{dx}$$

In terms of the local skin friction coefficient, $c_f = \tau_s / \frac{1}{2} \rho_a u_a^2$, using the value of ζ given in (3.14),

$$\Delta c_f = c_f - c_f^* = 2c \left(\frac{2-f}{f} \right) \frac{\lambda}{L} \left(\frac{dc_p}{ds} \right) \dots (4.14)$$

Now λ is the mean free path of the molecules at the wall, where the gas temperature is $T_w = T_{th}$; and so from (2.2),

$$c\lambda = \mu_s / \rho_s \bar{v}_s = \mu_a / \rho_s \bar{v}_a = \frac{p_a T_s}{p_s T_a} \left(\frac{u_a}{\rho_a \bar{v}_a} \right) \dots (4.15)$$

because*, within the accuracy of the kinetic theory of gases

* The implication is here that we treat $\mu \propto T^{\frac{1}{2}}$, which would seem to preclude the consideration of other conditions of the temperature variation of viscosity which we afterwards introduce. The point, however, is that the variation of μ cannot be accurately described by (2.2): in Sutherland's treatment of the problem, for instance, (see ref.1, §90),

$$\left(1 + \frac{S}{T_s} \right) \mu_s = c_1 \rho_s \bar{v}_s \lambda \propto T_s^{\frac{1}{2}}$$

where c_1 has at least a value close to that of c , and S is Sutherland's constant. Sutherland's formula is known to give a more accurate description of observed data, and from it we deduce

$$c\lambda = \left[\frac{c}{c_1} \left(1 + \frac{S}{T_a} \right) \right] \frac{p_a}{p_b} \frac{T_s}{T_a} \left(\frac{\mu_a}{\rho_a \bar{v}_a} \right)$$

which, apart from a difference in the constant of proportion, is the same as (4.15). Thus we shall regard (4.15) as definitive, in preference to (2.2), whatever the variation of μ with temperature, as it displays the essential fact that $(c\lambda)$ varies inversely with density: the constant of proportion in (4.15) may in fact be in error. If (4.15) is assumed, then, to be a definition, the assumptions implicit in its justifications are immaterial.

$\lambda \propto 1/\rho$ and so $\mu \propto \bar{v}$. The mean molecular speed is

$$\bar{v} = 2\sqrt{\frac{2p}{\pi\rho}} = 2\sqrt{\frac{2}{\pi\gamma}} a \quad \dots\dots\dots (4.16)$$

so that,

$$\frac{c\lambda}{L} = \sqrt{\frac{\gamma\pi}{8}} \frac{T_s}{T_a} \left(\frac{\mu_a}{\rho_a a L} \right) = \sqrt{\frac{\gamma\pi}{8}} \frac{M}{R} \frac{T_s}{T_a} \frac{p_a}{p_\delta} \quad \dots\dots (4.17)$$

Thus, from (4.17) in (4.14), since $T_s = T_{th}$,

$$\left. \begin{aligned} \Delta c_f &= \left(\frac{2-f}{f} \right) \sqrt{\frac{\gamma\pi}{2}} \frac{M}{R} \frac{p_a T_{th}}{p_\delta T_a} \left(\frac{dc_p}{ds} \right) \\ \text{or } R^{\frac{1}{2}} c_f &= (R^{\frac{1}{2}} c_f^*) + \left[\left(\frac{2-f}{f} \right) \sqrt{\frac{\gamma\pi}{2}} \left(\frac{p_a T_{th}}{p_\delta T_a} \right) \left(\frac{dc_p}{ds} \right) \right] \left(\frac{M^2}{R} \right)^{\frac{1}{2}} \end{aligned} \right\} (4.18)$$

In particular, we see that if the surface pressure gradient is zero then slipping has no effect on the skin friction; the correction in the presence of a velocity gradient becomes most important at high Mach number.

Similarly, if

$$Q = \left(k \frac{\partial T}{\partial y} \right)$$

is the rate of heat flux in the gas through the boundary layer,

$$Q_s = Q_s^* + \zeta(x) \left(\frac{\partial Q}{\partial y} \right)_s = 0 \quad \dots\dots\dots (4.19)$$

since, from (4.9), there is no heat flux into the wall. From (3.4)

$$\left(\frac{\partial Q}{\partial y} \right)_s = -\mu_s \left(\frac{\partial u}{\partial y} \right)_s^2 \quad \dots\dots\dots (4.20)$$

so that in (4.19), from (4.20), if $\Delta Q = Q_s - Q_s^*$, then

$$\Delta Q = -Q_s^* = -\mu_s \zeta \left(\frac{\partial u}{\partial y} \right)_s^2 = -\frac{\zeta}{\mu_s} (\tau_s^*)^2 \quad \dots\dots\dots (4.21)$$

In terms of the local heat transfer coefficient, $k_H = Q_s / \frac{1}{2} \rho_a u_a^3$, using (3.14)

$$\Delta k_H = -k_H^* = -c \left(\frac{2-f}{f} \right) \frac{\lambda}{L} \frac{\rho_a u_a L}{\mu_s} (c_f^*)^2$$

or from (4.17), if $\mu_s/\mu_a = (T_s/T_a)^\omega$,

$$\left. \begin{aligned} \Delta k_H &= -\left(\frac{2-f}{f}\right) \sqrt{\frac{\gamma \pi}{8}} \frac{p_a}{p_\delta} \left(\frac{T_{th}}{T_a}\right)^{1-\omega} M (c_f^*)^2 \\ \text{or } \frac{k_H^*}{c_f^*} &= \left(\frac{2-f}{f}\right) \sqrt{\frac{\gamma \pi}{8}} \frac{p_a}{p_\delta} \left(\frac{T_{th}}{T_a}\right)^{1-\omega} (R^{\frac{1}{2}} c_f^*) \sqrt{\frac{M^2}{R}} \end{aligned} \right\} \dots (4.22)$$

This expression shows that the heat transfer is zero when in fact the uncorrected value of the heat transfer coefficient is positive: i.e. for small rates of heat transfer, the effect of surface slip is to reduce the rate of heat transfer, most particularly at high Mach number. The effect is quite independent of the surface pressure gradient.

Finally, we note that the displacement thickness of the boundary layer is reduced, by slipping, by the distance ζ , i.e.

$$\begin{aligned} \delta_1 &= \int_0^\infty \left(1 - \frac{u}{u_a}\right) dy = \int_{-\zeta}^\infty \left(1 - \frac{u}{u_a}\right) dy - \int_{-\zeta}^0 \left(1 - \frac{u}{u_a}\right) dy \\ &= \delta_1^* - \left[\zeta - \int_{-\zeta}^0 \left(\frac{u}{u_a}\right) dy \right] \end{aligned}$$

But for $-\zeta < y < 0$, (u/u_a) is of order (λ/δ) so that

$$\Delta \delta_1 = \delta_1 - \delta_1^* = -\zeta \left[1 + O\left(\frac{\lambda}{\delta}\right) \right]$$

or, correct to the first order of approximation:

$$\delta_1 = \delta_1^* - \zeta = \delta_1^* - 2c \left(\frac{2-f}{f}\right) \lambda$$

Thus, from (4.17),

$$\left. \begin{aligned} \frac{\Delta \delta_1}{L} &= -\frac{2-f}{f} \sqrt{\frac{\gamma \pi}{2}} \left(\frac{p_a T_{th}}{p_\delta T_a}\right) \frac{M}{R} \\ \text{or } M \frac{\delta_1}{L} &= \left(R^{\frac{1}{2}} \frac{\delta_1^*}{L}\right) \left(\frac{M^2}{R}\right)^{\frac{1}{2}} - \left[\left(\frac{2-f}{f}\right) \sqrt{\frac{\gamma \pi}{2}} \left(\frac{p_a T_{th}}{p_\delta T_a}\right) \frac{M^2}{R}\right] \end{aligned} \right\} \dots (4.23)$$

4.3. The Correction in the Presence of Heat Transfer

As already remarked, the solution of the problem of the effects of surface slip and temperature jump on the flow of the boundary layer, with heat transfer to the surface, is more complicated.

/This is ...

This is because in the analogy with the no-slip flow about a surface depressed below the true surface, the temperature at this surface of no-slip is not simply the wall temperature, but differs from it by a certain amount (varying with x) as determined from equation (3.15).

The study of this problem involves the solution of the boundary layer equations of no-slip flow about a surface with varying surface temperature, and this problem only becomes tractable if some simplifying assumptions are introduced. Such additional assumptions are detailed below; even so, the results (which are also stated and discussed below) are a little involved, and their derivation is relegated to the Appendices II-IV.

4.31. The Solution for Heat Transfer in the Incompressible Boundary Layer

The term 'incompressible' applied to a boundary layer with heat transfer is rather inappropriate, but we mean by it the conditions obtaining if the Mach number of flight is small. If we further make the assumption that the change in temperature across the boundary layer is small, then the energy equation (3.4) becomes

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \left(\frac{k}{\rho c_p} \right) \frac{\partial^2 T}{\partial y^2} \quad \dots\dots\dots (4.24)$$

where u, v are sensibly the velocities in the layer if the temperature is everywhere constant, and $\left(\frac{k}{\rho c_p} \right)$ may be treated as a constant. An approximate integration of this equation has been suggested in ref. 5 by Fage and Falkner for the conditions in which

$$\left. \begin{array}{l} T_w = T_a + Bx^t \\ \text{and } u_0 = Cx^r \end{array} \right\} \quad \dots\dots\dots (4.25)$$

That is, with a power law variation for the wall temperature and surface pressure. Following the method of this reference, it is possible to deduce the correction in slip flow for these same conditions, on the basis of the analogy we have already formulated here. This is done in Appendix II. The result obtained is that the changes in the Nusselt Number or skin friction coefficient (i.e. the values of $\Delta(Nu) = Nu - Nu^*$, or $\Delta k_H = k_H - k_H^*$)

/are given ...

are given by the relation (II.6) of that Appendix, which is

$$\frac{\Delta(Nu)}{(Nu^*)} = \frac{\Delta k_H}{k_H^*} = - \sqrt{\frac{\gamma\pi}{2}} \left[\frac{F\left(\frac{r+t}{r+1} - \frac{1}{2}\right)}{F\left(\frac{t}{r+1}\right)} \right] \left[\left(\frac{2\gamma}{\gamma+1}\right) \left(\frac{2-a}{a\sigma}\right) - \left(\frac{2-f}{f}\right) \right] \\ \times \left(\frac{p_a}{p_\delta}\right) \left(\frac{Nu^*}{R^2}\right) \sqrt{\frac{M^2}{R}} \dots\dots\dots (4.26)$$

where F is a function, shown graphically in fig. 2. The factor

$$\left(\frac{2\gamma}{\gamma+1}\right) \left(\frac{2-a}{a\sigma}\right) - \left(\frac{2-f}{f}\right)$$

accounts for the difference between the temperature at the apparent surface of no-slip (T_ζ) and that of the wall (T_w). If it is zero, then $T_\zeta = T_w$, and there is no correction required to Nu or k_H for slip effects. In fact, it appears to be positive (see para. 4.4). On the other hand the sign of the correction also depends on Nu^* and $F(X)$: $F(X)$ is only positive for $X > -\frac{1}{2}$, in which range it is shown in fig. 2. Thus we may make certain qualitative deductions from equation (4.26).

(a) In particular, if the wall temperature and the velocity outside the boundary layer are constant ($r = t = 0$), then it follows from (4.26) and fig. 2 that the correction due to the temperature-jump is zero. This is an important result as it is the condition most nearly realised in practical examples.

(b) If the wall temperature is constant, but the velocity outside the boundary layer increases ($t = 0, r > 0$), the correction to Nusselt number is negative, implying a reduction in the heat transferred to or from the body. On the other hand, if the velocity decreases ($r < 0$), the correction is positive, implying the opposite conditions.

(c) Again, if the velocity outside the boundary layer is constant ($r = 0$), but the temperature decreases away from the nose along the surface, the correction to the heat transfer coefficient (i.e. Δk_H) is negative, implying an incremental heat transfer from the body to the air. If the temperature increases aft of the nose, the correction has the opposite sign, and there is either an increase in the heat transfer to the body or a decrease in that transferred to the air.

4.32. An Extension to the Compressible Boundary

To deal satisfactorily with the same kind of problem in compressible flow, (i.e. without introducing any assumption

/as to ...

as to the magnitude of M), it is necessary first to impose the condition that the flow is isobaric, i.e.

$$\frac{dp}{dx} = 0, \quad p_0 = p_a \quad \dots\dots\dots (4.27)$$

so that it is no longer possible as before to ascertain the interaction of changes in pressure and wall temperature as in incompressible flow. Further, we must assume constant specific heats and Prandtl number, and following the method of ref. 6 we deal with the flow properties if the viscosity varies linearly with temperature: i.e.

$$\omega = 1 \quad \dots\dots\dots (4.28)$$

in our previous notation. The argument is detailed in Appendix III. As a particular result of the above assumptions, it follows that the shear stress distribution is independent of the temperature distribution, so that its correction for slip is identical with that derived previously in para. 4.2. Then because of (4.27) above,

$$\Delta c_f = 0, \quad \text{and} \quad c_f = c_f^* = \frac{0.664}{R_x^{1/2}} \quad \dots\dots\dots (4.29)$$

If the wall temperature is expressed as a power series in x :

$$T_w = T_{th}^* + T_a \sum_t B_t s^t \quad \dots\dots\dots (4.30)$$

where T_{th}^* is the thermometer temperature in no-slip flow, it is shown in Appendix III that the correction to the heat transfer coefficient is given by

$$R_x^{1/2} \Delta k_H = -\sqrt{\frac{\gamma M^2}{2R}} \left\{ \left(\frac{2-f}{2f} \right) (R_x^{1/2} c_f^*)^2 + \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-a}{a\sigma} \right) - \left(\frac{2-f}{f} \right) \right] \Phi k_H^* Nu^* \right\} \quad \dots\dots\dots (4.31)$$

where Φ is a factor depending upon the temperature distribution of equation (4.30), and is given by

$$\Phi = \frac{\left[\sum_t F(t-\frac{1}{2}) F(t) B_t s^{t-1} \right] \left[\sum_t B_t s^t \right]}{\left[\sum_t F(t) B_t s^{t-\frac{1}{2}} \right]^2} \quad (4.32)$$

Hence F is approximately the same function as is shown in fig. 2; the difference arises only because of the approximations introduced in ref. 5 (from which the values of F were calculated) which were

/not introduced ...

not introduced in ref. 6 (from which the above result was derived).

The first term (depending on c_f^*) in (4.31) represents the effect of slip on the heat transfer, and the second (involving the accommodation factor α) represents the effect of the temperature-jump.

If $k_H = 0$, then k_H^* is small, and (4.31) is in agreement with the equation (4.22) already derived for this condition (if we put $p_a = p_0$ and $\omega = 1$, as given by equations (4.27) and (4.28)).

If M is small, $(k_H^* Nu^*)$ is a quantity of order $(1/M^2)$ and so is large compared with $(R^{\frac{1}{2}} c_f^*)^2$ which is of order unity. Thus in this so called 'incompressible flow' condition, the latter term may be neglected, and from (4.31)

$$\frac{\Delta k_H}{k_H^*} = -\sqrt{\frac{\gamma\alpha}{2}} \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha\sigma} \right) - \left(\frac{2-f}{f} \right) \right] \overline{\Phi} \left(\frac{Nu^*}{R^{\frac{1}{2}}} \right) \sqrt{\frac{M^2}{R}}$$

Noticing that for a temperature distribution of the type

$$T_w = T_{th}^* + T_a B_t s^t = T_a (1 + B_t s^t) \quad \text{if } M = 0,$$

then from (4.32)

$$\overline{\Phi} = F(t - \frac{1}{2}) / F(t)$$

It will be seen that the new result is quite compatible with that of (4.26) previously obtained for the incompressible flow condition, if $p_a = p_0$ (and so $r = 0$).

If the surface temperature is uniform (whatever the Mach number), in (4.32) we have that $t = 0$, and since $F(-\frac{1}{2}) = 0$ it also follows that $\overline{\Phi} = 0$. Thus, for $T_w = \text{const}$, using (4.24)

$$R^{\frac{1}{2}} \Delta k_H = -\sqrt{\frac{\gamma M^2}{2R}} \left(\frac{2-f}{2f} \right) (R^{\frac{1}{2}} c_f^*)^2 = -\frac{0.33}{s} \left(\frac{M}{R^{\frac{1}{2}}} \right) \dots\dots\dots (4.33)$$

which is an important relation, since in most applications the temperature of the wall at least does not greatly vary.

If it were to vary we should have, in accordance with (4.31), to add an increment to Δk_H , which (as in the previously discussed condition of incompressible flow) is positive or negative, according to whether the wall temperature is (in general) either increasing or decreasing downstream. In the usual flight condition the skin temperature is highest near the nose, so that the correction (4.33) would slightly underestimate the reduction in the heat

/transfer ...

transfer to the skin resulting from considerations of the finite molecular free path.

Because we can construct the correction to the heat transfer coefficient resulting with a given skin temperature distribution, it should be possible to solve the inverse problem and in particular to establish the correct value of T_{th} , the thermometer temperature, (i.e. the temperature of the wall in the condition of zero heat-transfer). It would be expected that this value would differ only little from its value T_{th}^* in no-slip flow. However, as is fully discussed in the Appendices III and IV, this condition involves a singularity in the mathematical solution, and the known boundary conditions do not provide us with the information necessary to determine the precise difference between T_{th} and T_{th}^* . However, it is found that the 'equilibrium' condition of zero heat transfer can only be reached in slip flow after heat has been lost from the boundary layer to the external flow; or, in other words, the thermal layer then extends outside the velocity layer. More precisely, whereas in no-slip flow the temperature decreases exponentially as the square of the distance from the surface, in slip flow (in the condition of zero-heat transfer) it decreases merely inversely with this distance. The physical interpretation of this result is obscure: it would suggest that the equilibrium condition would not be reached as quickly in the presence of slip as otherwise, because an infinite flux of heat from the boundary layer to the air is necessary to attain this condition. On the other hand, our approximations may not be adequate for the treatment of this particular condition, the true value of T_{th} being dependent on conditions near the nose where our theory breaks down.

4.4. The Accommodation & Momentum Transfer Coefficients

The numerical evaluation of the corrections to be applied to the skin friction and heat transfer coefficients depends on a knowledge of the coefficients α and f . Insufficient experimental information exists to make more than a rough approximation to these values.

Laboratory tests suggest the following results for the value of the accommodation coefficient:

(i) α increases with the molecular weight of the gas (e.g. from 0.2 for H_2 up to 0.8 for N_2 or O_2 on a platinum surface)

/(ii) ...

(ii) α increases with surface temperature (e.g. for H_2 on Pt, from 0.2 to 0.5 as the temperature is raised to $1000^\circ C$ from room temperature)

(iii) α depends on the surface finish or quality (e.g. α rises from 0.3 - 0.4 up to 0.7 - 0.9 if a platinum surface is blackened with platinum black)

(iv) α depends on the age and history of the solid boundary (e.g. although α is recorded as 0.2 when a fresh gas-filmed surface is tested, the value $\alpha = 0.8$ is obtained for an old, but otherwise similar, specimen surface).

(v) α depends upon the material of the solid surface. The following values are taken from ref. 7 and refer to air on:

cast iron, polished	$\alpha = 0.87 - 0.93$
machined	$\alpha = 0.87 - 0.88$
etched	$\alpha = 0.89 - 0.96$
aluminium, polished	$\alpha = 0.87 - 0.95$
machined	$\alpha = 0.95 - 0.97$

Results for the value of f , the momentum transfer coefficient, display similar trends, and in every case where a comparison is possible, it is found that f is significantly larger than α , though likewise it has never been found to exceed unity: (e.g. for hydrogen on glass $\alpha = 0.36$ but $f = 0.89$).

All this evidence suggests that for anything other than fresh, clean, cold 'laboratory specimens' the value of α is at least 0.7 and such a figure corresponds only to the results of tests using light gases. On the kind of manufactured surface which an aircraft might possess, in the presence of air possibly supplying heat to it, we might therefore expect that α would be between 0.9 and 1.0, and that f would be virtually unity.

The value of σ is also required to evaluate the corrections, and we have that $\sigma = \frac{14}{19}$, $\gamma = \frac{7}{5}$, for air, (approx.). It occurs in combination with α and γ as the expression

$$\frac{2\gamma}{\gamma+1} \left(\frac{2-\alpha}{\alpha \sigma} \right)$$

which, for air, has the value 1.75 if $\alpha = 0.95$. We would therefore be safe in assuming that

$$\left[\frac{2\gamma}{\gamma+1} \left(\frac{2-\alpha}{\alpha \sigma} \right) - \frac{2-f}{f} \right] > 0.$$

and it follows that, from (3.15), in the presence of heat transfer to the wall, the value of T_g (the temperature of the air at the

/plane of ...

plane of 'no-slip' below the true surface) exceeds the wall temperature, as also does the temperature of the air at the wall.

4.5. Limitations and Applicability of Solutions

In the preceding paragraphs we have obtained precise expressions for the corrections to be applied to k_H^* and c_f^* for the compressible boundary layer with no heat transfer; the case of the compressible boundary layer with heat transfer but without a surface pressure gradient is also considered, subject to the assumptions that the Prandtl Number is unity and that the viscosity is proportional to the absolute temperature. The last mentioned assumptions are not essential, as we have seen, only if $M = 0$, but they are probably sufficiently accurate for most conditions encountered in subsonic flow.

We have not been able to derive the corrections to be applied in the condition of heat transfer at high Mach number; nor have we been able to deal with the condition of heat transfer in the presence of a surface pressure gradient, except for the condition that $M = 0$.

Presumably some of these problems could be solved by introducing approximations, but this will not be attempted here. The difficulty lies in the variation of the temperature-jump along the surface: the assumption of a mean constant temperature jump is unsatisfactory, as we have seen that the form of the correction to be evaluated depends critically on the form of the variation of temperature-jump.

However, we have considered a fairly wide range of conditions and it is important at this stage to examine the applicability of the solutions. The corrections we have evaluated do not enable us to determine either the total skin friction or the total heat flux to the surface. This point may best be understood by considering the correction to the rate of heat transfer. It is made up of two terms, in general: the first depending on the effects of slip, which varies as $(c_f^*)^2$, and the second on the effects of the temperature-jump varying as $(k_H^*)^2$. The first term (if not the second, also) varies as $1/x$, so that although it is in general small, it is not so, near the origin of the boundary layer. In fact as $x \rightarrow 0$, the correction exceeds the value of the uncorrected heat transfer coefficient. This is plainly a contravention of our initial assumptions that the correction due to slip is small, and it arises because the boundary

/layer equations ...

layer equations are no longer valid near the origin of the layer. An examination of the correction terms shows them to be of order $\sqrt{M^2/R_x}$ compared with the uncorrected values, and evidently they are not small (as assumed) if R_x becomes a magnitude of order M^2 . In such a condition the approximations involved by our analysis become invalid. Strictly, then, our results apply only if

$$\left. \begin{array}{l} R_x \gg M^2 \quad \text{i.e.} \quad \frac{x}{L} \gg \frac{M^2}{R} \\ \text{or} \quad x \gg \frac{\mu_a u_a}{p_a} \quad \text{i.e.} \quad \frac{x}{L} \gg M \frac{\lambda}{L} \end{array} \right\} \dots\dots\dots (4.34)$$

The restriction is unimportant in relation to the values of the rate of heat flux, since usually such values are not needed near the origin of the boundary layer, and the local rate of heat transfer downstream of the nose is a quantity which has greater significance in boundary layer theory.

However, the local rate of shear stress is rarely a criterion of much significance: the total skin friction force is the property which is required to be known, and in this connection the present analysis is of little help. Even if downstream the correction to the local shear stress is zero - as is the case, as we have seen, if the pressure gradient is zero - we may not infer that this is also true near the nose. In fact, certainly the shear stress must be finite near the nose, since the momentum carried to the surface by the impinging molecules must be finite. On the lines of the kinetic theory of gases, the number of impinging molecules may be calculated as $\frac{1}{4} N \bar{v}$ (in the notation of Appendix I) and each carries at most a sideways momentum mU , where U is some velocity which will be commensurate with that of the surface relative to the air. The maximum transfer of momentum is thus

$$\frac{1}{4} N \bar{v} mU = \frac{1}{4} \rho \bar{v} U = \frac{1}{4c} \frac{\mu U}{\lambda}$$

or in other words,

$$(c_f)_{\max} = 0 \left(\frac{\mu_a}{\rho_a u_a \lambda} \right) = 0 \left(\frac{1}{M} \right) = \frac{K}{M}, \text{ say.}$$

Hence, if c_f is the total skin friction and c_f^* is its value uncorrected for slip effects

$$c_f - c_f^* < \int_0^s [(c_f)_{\max} - c_f^*] ds$$

/if ...

if $c_f^* = (c_f)_{\max}$ at $s = \varepsilon$. But in general we find that, in the absence of a pressure gradient, $R^{\frac{1}{2}} c_f^* = (R_x^{\frac{1}{2}} c_f^*) s^{-\frac{1}{2}}$ where $(R_x^{\frac{1}{2}} c_f^*)$ has a finite constant value given by equation (4.29). Hence, we calculate that

$$\varepsilon = \left(\frac{R_x^{\frac{1}{2}} c_f^*}{K} \right)^2 \frac{M^2}{R},$$

and so

$$R^{\frac{1}{2}} (c_f - c_f^*) < - (R_x^{\frac{1}{2}} c_f^*)^2 \frac{M}{KR^2} \dots\dots\dots (4.35)$$

Thus there will be a first order negative correction term to C_f in $\sqrt{M^2/R}$, i.e. one which is commensurate with those we have been investigating by the present discussion, - and such a correction is due to the modification to the boundary layer flow near the nose, and not at positions downstream.

Practically, then, the use of the results we have derived here, to account for slip effects on the shear stress at the surface downstream of the nose, will be to enable a check to be made of more approximate theories which may account for the changes near the nose as well as downstream. This we shall therefore attempt to do in the next paragraph.

The results for the correction to the heat transfer coefficient are, on the other hand, more likely to be of some practical value. In this connection it is worthwhile noting the conditions of flow likely to necessitate use of the correction. Our corrections are in general (for finite Mach numbers) of the order of $(1/R_x^{\frac{1}{2}})$, as already noted: and we have also noticed that there are other corrections, which we have neglected, of the order of $(1/R_x)$. If then we are satisfied with a numerical estimate accurate to within 1 per cent, the corrections would be inappreciable if the local Reynolds number exceeded about 10^4 , and inaccurate (within the stipulated margin) if the local Reynolds number were below 10^2 . On the other hand, if we were satisfied with a wider assessment (with some 10 per cent margin of error) then it would be legitimate to apply the corrections down to Reynolds numbers of 10. In making these suggestions, it must be remembered that there is a great danger in interpreting magnitudes, which are assumed (mathematically) infinitesimal, as numbers. The presence of a numerical factor (of 0.1 or 10, say) to be applied to such a magnitude would of course make no difference to the theoretical reasons for neglecting or including it, but nevertheless would make a great difference to its numerical importance in calculations.

5. Comparison with Other Theoretical Results

The problem of slip flow in relation to the boundary layer has so far attracted little attention.[‡] In a short note,⁸ Schaaf has approached the problem of the effect of surface slip on the skin friction of an incompressible boundary layer over a semi-infinite flat plate using Raleigh's method - i.e. by considering the one-dimensional growth of a laminar layer with time. The method is an elegant and simple one and yields a value of the local skin friction coefficient c_f , which may be expressed as the asymptotic series valid if $R_x \gg M^2$,

$$R_x^{\frac{1}{2}} c_f = 0.664 \left[1 - 0.380 \frac{M^2}{R_x} + \dots \right] \dots\dots\dots (5.1)$$

This was shown in ref. 9 which contains a generalisation of Schaaf's results to compressible flow on the basis of the assumptions we made here in para. 4.32. Because there is no term in the right hand side of (5.1) in $\sqrt{M^2/R_x}$, it is evident that this result is in agreement with the deductions we have made here, - that there is no first-order effect of slip on the local rate of skin friction. The term in (M^2/R_x) we would not find by the method outlined in the present report as terms of such order are the error terms in the original boundary layer equations. Having in mind the fact that no account is taken of the non-linear terms in viscosity in the equations of ref.8 as well, it is open to doubt whether the results of this reference have any greater significance than those of this report. However, as is suggested in ref. 9, the results might be a guide to the conditions existing even if the assumptions are inaccurate, and it is noteworthy that the answer obtained for the value of c_f in free-molecule flow (where $R_x \ll M^2$) is only twice that developed more accurately by Ashley in ref. 10. Hence the results do not appear to be too inaccurate and may be applied to calculate the flow near the origin of the boundary layer so as to give the value of

$$R_x^{\frac{1}{2}} c_f = 1.328 \left[1 - 0.773 \frac{M}{R_x^{\frac{1}{2}}} + 0.380 \frac{M^2}{R_x} + \dots \right] \dots\dots\dots (5.2)$$

As noticed in para. 4.5, a first-order term in $(M/R_x^{\frac{1}{2}})$ arises due to the behaviour of the flow near $x = 0$. No doubt inclusion

/of other

‡ A recent discussion¹² of the effects of slip on the skin friction on a flat plate in incompressible flow has included the particular results of the present work: (a) that $\Delta c_f = 0$ to a first approximation, and (b) that there is a reduction in the displacement thickness.

of other terms in the equations of motion might enable a more accurate answer to be obtained; but it must be admitted as unlikely that any single solution on such lines could accurately describe at the same time both types of free-molecule and continuum flow. An approximate treatment must be undertaken, and Schaaf's results seem adequate.

The expression of $(R^{\frac{1}{2}}c_f)$ in terms of powers of $(M/R^{\frac{1}{2}})$ in equation (5.1), bears out the assertion, made originally in ref. 4, that it is this parameter which is the important criterion of 'superaerodynamic' phenomena. This is also borne out by the present results, as will be seen from an examination of the formulae for $R^{\frac{1}{2}}c_f$ and $R^{\frac{1}{2}}k_H$ in equations (4.18), (4.22), (4.26) and (4.31). However, it will be observed that the coefficients of the powers of $\sqrt{M^2/R}$ in these equations, whilst in general finite for $M = 0$, do in fact vary with Mach number, so that for large values of M it may be some other parameter involving M and R which is important. Thus, as shown in ref. 3, the significant parameter for slip effects if $M^2 \gg 1$, is in fact, $\sqrt{M/R}$. However, this is not a very important difference.

The third-order Boltzmann Equation has been used by Schamberg⁷ to establish the effects of the inhomogeneity of the air on the Couette flow,[‡] and by the inclusion of the non-linear terms in viscosity and heat conduction, a much more accurate treatment may be made than that attempted here. On the basis of these results - which show the well-established decrease in shear stress with slipping, - Schamberg suggests that there is also a decrease in skin friction of a similar magnitude due to the slipping of the flow in the boundary layer. On this basis he suggests, in fact, that

$$\frac{c_f}{c_f^*} = 1 - 0.81 \frac{M}{R_x^{\frac{1}{2}}} + (0.66 + 0.10 M^2) \frac{M^2}{R_x} \dots\dots\dots (5.3)$$

which indicates a first-order reduction in c_f due to slip. This result follows from Schamberg's work, however, because he assumes that the boundary layer thickness is unchanged: we have suggested that the boundary layer thickness is actually reduced by slipping.

To emphasise this point we note that in a Couette flow slipping at either of the two plates must produce a decrease in the velocity gradient in the flow between them, and so also a reduction in skin friction. However, in a boundary layer flow there is no second surface apart from the wall: if we must imagine it to have a finite thickness (a concept we have not elsewhere envisaged) then the results of this note suggest that,

/although ...

‡ i.e. a one-dimensional shear between parallel walls.

although the air slips at the surface, the boundary layer thickness is reduced (by a distance we have termed ζ), so that there is not necessarily a decrease in the velocity gradient across the boundary layer. (Indeed, under certain circumstances we have seen that there may even be an increase in skin friction). Furthermore, to draw an analogy between Couette flow and boundary layer flow, would also imply that there is slipping of the air at the outside of the boundary layer which is plainly not so since there is no shear between the 'outside' of the boundary layer and the external flow.

These points are important, not because in any way they detract from Schanberg's remarkable paper, which has as its main theme a different problem from the one discussed here, but because it illustrates what is apparently a mistaken analogy, which one is tempted to use in describing slip effects in a boundary layer.

The effect of slip flow on heat transfer has been considered by Drake and Kane,¹¹ and here although there is no precise agreement with the present results, a decrease is predicted in the rate of heat transfer due to the temperature jump: however, as the results when applied to the continuum regime ($M^2/R \ll 1$) yield a value for the rate of heat transfer which is four times as large as that derived from the usual boundary layer theory, a numerical comparison with the results of the present work could hardly be expected to provide more than qualitative agreement. Like Schaaf's work on the shear stress, this work sets out to include the full range of flow conditions - from continuum to free-molecule flow.

6. Conclusions

(i) We have developed in this report a method of finding the first-order effects of the existence of a small but finite mean free-path of the air molecules on the local rate of shear stress and heat transfer at a surface in the presence of a laminar boundary layer.

(ii) The results are applicable to any type of two-dimensional surface, but it should be noted that if the surface is curved, error terms arise in the expression of the values of c_f and k_H due to the surface curvature which are generally neglected in the usual treatment of the boundary layer theory, but which are in fact commensurate with the other corrections we derive here.

(iii) The method adopted is to express c_f and k_H as a power series in the parameter $(1/R^{1/2})$. The first term in this

/series is ...

series is the value usually found from boundary layer theory assuming the air to be a continuous medium; the second term in the series is (if the surface is plane) the first-order effect due to surface slip and temperature-jump, and is the term we derive here. The third and higher terms denote the error in our treatment: such terms can only accurately be derived if we relax the approximations used by Prandtl to describe the equations of a thin boundary layer, and if we include the non-linear terms in viscosity and conductivity in the statement of the equations of momentum and energy.

(iv) Due to the singular behaviour of the properties of the boundary layer near its origin the assumptions implicit in the construction of Prandtl's boundary layer equations (and in the present analysis) do not apply at positions on the surface where the local Reynolds number is no longer high. Thus our results are applicable only as corrections to the shear stress and heat transfer downstream of the nose. This means that we cannot estimate the total correction to the skin friction drag, and a qualitative examination of the flow conditions near the origin of the leading edge shows that, in this region, there is a correction to be applied to the total intensity of skin friction (as usually derived from boundary layer theory) which is commensurate with the total correction due to slip downstream of the nose.

(v) In the practical applications of the knowledge of the heat transfer from a boundary layer, it is usually the local - as distinct from the total - rate of heat flux which must be evaluated, and the results we derive here are of use in estimating this local value except at positions within a distance from the nose commensurate with the molecular mean free path (see the expressions (4.34)).

(vi) It is shown that, within the accuracy of our solution, the expressions derived from the kinetic theory of gases for the slip velocity of a gas (in a simple shearing motion) at a surface, and the temperature-jump at a surface exposed to stationary gas (through which heat is being conducted), are valid first-approximations to the slip velocity and temperature-jump at a surface within a boundary layer.

(vii) The values of the slip velocity and temperature-jump are expressed in terms of two coefficients (the coefficient of momentum transfer, f , and the accommodation coefficient, α) whose precise value must be obtained from experiment. Some further remarks on their values are given in para 4.5.

/(viii) ...

(viii) Given the corrected forms of the boundary conditions in velocity and temperature at the surface, it is then shown that the boundary layer equations yield a solution (correct to the order of approximation used) which implies that the flow is the same as if, in place of the actual surface boundary conditions, the usual condition of no-slip were applied at a boundary slightly below the true surface (see fig. 1). The boundary layer flow over this apparent 'surface of no-slip' may be calculated by the usual methods, and the conditions at the position corresponding to that of the true surface are then the corrected surface conditions. The depression of this surface of no-slip below the true one is

$$\zeta = 2 \left(\frac{2-f}{f} \right) c\lambda = \sqrt{\frac{\gamma\pi}{2}} \left(\frac{2-f}{f} \right) \left(\frac{p_a T_s}{p_\delta T_a} \right) \frac{M}{R} L$$

and is evidently a distance of the order of the mean free path.

(ix) The exploitation of this method of analysis is complicated by the fact that to account properly for the effect of temperature-jump the temperature at the surface of no-slip, T_ζ , has to be different from that of the wall, T_w , and this temperature difference ($T_\zeta - T_w$) varies along the surface since it is dependent on the wall temperature gradient (as expressed in equation (4.15)).

(x) The temperature-jump is zero if there is no heat transfer, and then it is calculable that the reduction in the local skin friction coefficient due to slip is

$$\Delta c_f = \left(\frac{2-f}{f} \right) \sqrt{\frac{\gamma\pi}{2}} \frac{M}{R} \left(\frac{p_a T_{th}}{p_\delta T_a} \right) \frac{dc_p}{ds}$$

In particular, this correction is zero if the pressure gradient $dc_p/ds = 0$: since (in general) the pressure decreases over aerodynamic surfaces backwards from the stagnation point, it follows that slip usually provides a reduction in the shear stress intensity.

(xi) For small rates of heat transfer the correction to the local heat transfer coefficient, likewise, is due entirely to the effects of surface slip. It is given by

$$\Delta k_H = - \left(\frac{2-f}{f} \right) \sqrt{\frac{\gamma\pi}{8}} \frac{p_a}{p_\delta} \left(\frac{T_{th}}{T_a} \right)^{1-\omega} M(c_f^*)^2$$

where c_f^* is the uncorrected skin friction coefficient, and the correction is essentially negative, denoting either a reduction in the heat transfer to the skin, or an increase in that transferred to the air.

/(xii) ...

(xii) As a corollary of this last result, it follows that the true heat transfer rate is zero when the uncorrected value is small, but positive. This would seem to indicate a reduction in the value of the thermometer temperature: such a reduction is shown to be dependent on the conditions existing near the nose, and so it cannot be assessed by the present analysis. But it may also be shown that the effects of slip in this condition imply a transfer of heat to the flow outside the boundary layer. An infinite heat flux is needed to attain this equilibrium condition, so that we would infer that the slip effects, in practice, would retard the development of an equilibrium state of zero heat transfer.

(xiii) The effects of the temperature-jump on the rate of heat transfer may be assessed for incompressible flow ($M = 0$): a generalised form of equation (4.26) shows that if the surface temperature is given by T_w where

$$T_w - T_{th} = T_a \left(\sum_t B_t s^t \right) \quad (s = x/l)$$

then the correction to be applied to the uncorrected heat transfer coefficient (k_H^*) is given by

$$k_H = -\sqrt{\frac{\gamma M^2}{2R}} \left(\frac{p_a}{p_\delta} \right) \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha\sigma} \right) - \left(\frac{2-f}{f} \right) \right] \bar{\Phi} k_H^* Nu^*$$

where Nu^* is the uncorrected Nusselt number; here $\bar{\Phi}$ is a function of s defined by

$$\bar{\Phi} = \left[\sum_t F \left(\frac{r+t}{r+1} - \frac{1}{2} \right) F \left(\frac{t}{r+1} \right) B_t s^{t-1} \right] \left[\sum_t B_t s^t \right] / \left[\sum_t F \left(\frac{t}{r+1} \right) B_t s^{t-\frac{1}{2}} \right]^2$$

the value of r being defined by the assumption that the velocity outside the boundary layer $u_\delta \propto s^r$, and the function F being shown in fig. 2. In particular, if the surface temperature is uniform, it follows that

$$\bar{\Phi} = 2.04 F \left[-\frac{1-2r}{2(1+r)} \right]$$

and further if there is no surface pressure gradient ($r = 0$), then $\bar{\Phi} = 0$: thus with no change in pressure or temperature along the surface, the correction due to the temperature-jump is zero. An increasing pressure downstream tends to increase the transport of heat to or from the body: a decreasing pressure, to reduce it. On the other hand, an increase of the surface temperature along the surface implies a correction indicating a transport of heat towards the body: and a decreasing temperature,

/a transport ...

a transport of heat away from the body. Thus, in the condition most often met in practical examples - with heat transfer to the wall, and the pressure and surface temperature decreasing downstream - the heat transfer to the body is reduced by the effects of the temperature-jump.

(xiv) The problem of the corrections to be applied to the conditions in the compressible boundary layer has only been ascertained exactly in the absence of a pressure gradient, and assuming that the gas viscosity varies in proportion to the absolute temperature. Then it is found that the skin friction intensity is unaffected by the temperature-jump, and as (for zero pressure gradient) the correction due to slip is also zero (as shown in (x)), it follows that the uncorrected value c_f^* is also accurate within the limits of the approximation used in this analysis: i.e.

$$c_f = c_f^* \left[1 + O\left(\frac{1}{R}\right) \right]$$

The correction to the heat transfer coefficient (Δk_H) is found to be simply the algebraic sum of the corrections due to slip (for zero heat transfer) and due to temperature-jump (in incompressible flow) as given above in paragraphs (xi) and (xiii). In particular, if the surface temperature is uniform the latter correction is zero and (taking $f = 1$)

$$k_H = k_H^* - 0.33 \frac{M}{R_x}$$

(xv) As a rough indication, the corrections mentioned above are likely to be numerically important if the local Reynolds number is between about 10^2 and 10^4 . For a wider estimate the corrections may be applicable if the local Reynolds number is rather lower, but greater than about 10.

(xvi) In refs. 8 and 9 an attempt has been made to estimate the effects of slip on the structure of the boundary layer, using the same assumptions as in the present report (and for conditions as given in the last paragraph) but applying them by Raleigh's method of solution. The result for the value of c_f is compatible with that stated in (xiv) above: the method happens to be also applicable even near the leading-edge of the surface, but it is doubtful whether there the solution has much quantitative significance. The assumptions certainly do not justify any further rigorous extension of the results beyond those included in the present analysis. This matter is discussed, at length, in para. 5 where a similar essay on the

heat transfer problem is noted.

(xvii) An alternative approach to the problem is by analogy with the simple Couette flow between parallel plates. This is exemplified in ref.7, but as is shown in para. 5, the analogy is mistaken since in the Couette flow the depth of the disturbed flow is fixed by the distance between the plates and slipping must necessarily reduce the velocity gradient; however, in boundary layer flow, there is no constraint of a second surface and the layer 'thickness' is actually decreased by slipping (by the distance ζ) so that there is not necessarily, as we have seen, any reduction in the velocity gradient at the wall.

(xviii) Further work on the lines of the present analysis might increase its applicability (to a more complete examination of the problem of compressible flow) or its accuracy (in the assessment of higher-order effects) but the main need is for some method of finding the true conditions of flow near the origin of the boundary layer, since without this knowledge we cannot consider the overall effects of the non-uniformity of the air flow.

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APPENDIX I: THE PHYSICAL THEORY OF SLIP
AND THE TEMPERATURE JUMP

Although the physical problems associated with the concepts of viscosity and heat conduction as derived from the kinetic theory of gases are fully dealt with in text-books (e.g. refs. 1 and 2) it has been thought worthwhile here briefly to restate the basic ideas, since they are fundamental to the treatment used in the body of this note.

We are concerned with the processes of heat conduction and viscous traction within the gas and at a surface. We shall first consider the latter problem - that of the mechanism of the viscous forces - as an example of the mathematical formulation of the ideas which lead to the concept of slip phenomena.

Considering the motion of a continuous gas, we may define the viscosity of the medium in relation to the shearing component of the forces acting on one side of an element of surface moving with the gas. Thus if δS is a small surface element moving with, and tangential to, a stream of gas flowing everywhere in the same direction (i.e. a simple shear flow) then we may define a coefficient of viscosity μ , by writing the force on the element

$$\delta F = \mu \frac{\partial q}{\partial n} \delta S \quad \dots\dots\dots (I.1)$$

where $(\partial q / \partial n)$ is the velocity gradient in the gas in the direction of the outward normal to the surface. Defined in this manner, μ is found from experiment to be a quantity dependent only on the nature and state of the gas.

We define the shear stress at the surface element as $\tau = \lim_{\delta S \rightarrow 0} (\delta F / \delta S)$, i.e.

$$\tau = \mu \frac{\partial q}{\partial n} \quad \dots\dots\dots (I.2)$$

and if there is no acceleration on the surface element the shear stress on the two surfaces of the surface will be equal and opposite, and $\frac{\partial q}{\partial n}$ will be continuous in the gas.

If the gas is assumed a discontinuous medium - that is, it is assumed constructed of molecules - then the shear stress on a solid surface may be related to the transfer of momentum between the molecules and the surface. However, although in a homogeneous flow we may define the shear stress

/inside the ...

inside the gas by (I.2), a similar concept is more difficult to envisage in molecular flow in the absence of solid boundaries. Suppose we consider the conditions at a plane drawn within such a flow such that its tangent is parallel to, and the plane moves with, the local mean stream velocity: we may, without loss of generality, consider this plane to be at rest. Then over a period of time, molecules will be passing through this plane, carrying from one side to the other varying amounts of sideways momentum. The mean momentum of all passing molecules is, in fact, zero by definition, so that if from one side, due to the variation in stream velocity, the entering molecules have on average a certain positive momentum, then those entering from the opposite side must carry, on average, an equal but opposite momentum. There will be, as a result, a transfer of momentum from one side of the plane to the other, which we may liken to the existence of a shear stress on the plane. A comparison of this with the definition of μ allows us to interpret its value in terms of molecular dimensions, and we find in fact that

$$\mu = c\bar{v}\lambda \quad \dots\dots\dots (I.3)$$

where \bar{v} is the mean molecular speed, λ the mean free path and c is a numerical constant.

The relation is in this form since the momentum brought up to the plane is proportional to

$$m \frac{\partial q}{\partial n} \lambda$$

where m is the molecular mass, and the molecules arrive with a velocity characteristic of that at their point of previous collision; whereas the number crossing the plane is proportional to $N\bar{v}$ per unit area per unit time (N being the number of molecules per unit volume). Since $\rho = Nm$, the result of (I.3) follows by comparison with (I.2).

If we now consider the transfer of momentum to a solid surface in a mean motion which is a simple shear, we note that the impinging molecules bring up to the surface the same momentum as that which is brought up to one side of a plane in the interior of the gas. This is precisely half the momentum which is being transported, in a simple shearing motion, through planes parallel to the surface within the gas. In order that the momentum transfer should be continuous, it follows either that the molecules leaving the surface do so with their tangential components of velocity just reversed, or else that the molecules have a mean motion relative to the surface. The

/former ...

former alternative is very unlikely, and we may generally assume that, on striking the surface and thereafter departing, the molecules give to it (on average) a certain fraction f of their momentum. If the molecules have no mean motion relative to the surface, it would be necessary that $f = 2$; however f , generally termed the 'momentum transfer coefficient', is found to be usually rather less than unity. Accordingly, we must envisage the existence of a 'velocity of slip' of the mean motion of the molecules of the gas at the wall.

Thus, the amount of momentum brought to the surface is simply

$$\frac{1}{2} \mu \frac{dq}{dn} + \frac{1}{4} N\bar{v} m u_s$$

where u_s is the 'slip velocity' at the wall, since $\frac{1}{4} N\bar{v}$ represents the number of molecules incident per unit time on unit area of surface. Writing $Nm = \rho$, and equating the momentum given up to the surface to that transmitted across parallel planes in the gas, we find that

$$f \left(\frac{1}{2} \mu \frac{dq}{dn} + \frac{1}{4} \rho \bar{v} u_s \right) = \mu \frac{dq}{dn} \quad \dots\dots\dots (I.4)$$

Hence u_s must have the value, using (I.3), of

$$u_s = 2c \left(\frac{2-f}{f} \right) \frac{dq}{dn} \lambda \quad \dots\dots\dots (I.5)$$

We now define a 'coefficient of slip'

$$\zeta = \frac{u_s}{dq/dn} \quad \dots\dots\dots (I.6)$$

which may be envisaged as that distance below the surface where the mean gas velocity would be reduced to rest relative to the true surface. From (I.6) and (I.5), this apparent depression of the plane of 'zero slip' is

$$\zeta = 2c \left(\frac{2-f}{f} \right) \lambda \quad \dots\dots\dots (I.7)$$

This treatment is due mainly to the original work of Maxwell. Another approach to the subject is suggested by Chapman and Cowling²: if q_1 is the component of velocity tangential to the plate of the molecules before striking, and q_2 is its value after reflection, then the mean gas velocity at the plate is simply $\frac{1}{2}(q_1 + q_2)$ since as many molecules must reach the plate as leave it. If a fraction f of all the

/molecules ...

molecules reflect specularly, carrying away the same sideways momentum as they had before striking, and the remainder are trapped in the surface and re-enter the gas with a random velocity - so that they have on the average the same velocity (q_w) on leaving as that of the plate - then

$$q_2 = (1-f)q_1 + fq_w.$$

But on the other hand, the impinging molecules have a sideways momentum which is representative of that at a distance of the order of magnitude of the mean free path from the plate, so that we may put

$$q_1 = \frac{1}{2}(q_1 + q_2) - c\lambda \frac{dq}{dn}$$

where c is called the 'persistance' factor. Hence the velocity of slip is simply the difference between the velocity of the wall and the mean gas velocity at the wall: i.e.

$$u_s = q_w - \frac{q_1 + q_2}{2} = \frac{2(2-f)}{f} c\lambda \frac{dq}{dn}$$

which is identical with (I.5). The relation of c to its previous definition in (I.3) does not follow from this analysis.

An exactly similar argument to that of Maxwell's may be followed in interpreting the processes of heat conduction within an inhomogeneous gas. If the gas acts as a continuum we may define the heat flux as

$$Q = k \frac{\partial T}{\partial n} \dots\dots\dots (I.8)$$

and k , the thermal conductivity may be interpreted as

$$k = \gamma c_v \mu / \sigma \dots\dots\dots (I.9)$$

where γ/σ is some constant. In a motion in which the heat flux is a constant throughout the gas, we infer likewise that there must be a temperature difference between the gas at the surface and the surface itself, which we shall write as $(T_s - T_w)$. Knudsen defines an 'accommodation coefficient':

$$\alpha = \frac{E_i - E_r}{E_i - E_w} \dots\dots\dots (I.10)$$

where E_i is the energy brought up to unit area per unit time by the incident stream, E_r that carried away by these molecules

/as they ...

as they leave the wall after reflection from it, and E_w is the energy that this latter stream would carry away with it if it carried the same mean energy per molecule as does a stream issuing from a gas in equilibrium at the wall temperature T_w ; then equating the heat flux through the gas with that into the surface

$$c \left[\frac{1}{2} k \frac{\partial T}{\partial n} + \frac{1}{4} \rho \bar{v} \frac{\gamma+1}{2} c_v (T_s - T_w) \right] = k \frac{dT}{dn}$$

where γ is the ratio of the specific heats of the gas; so that

$$T_s - T_w = \frac{4\gamma}{\gamma+1} \frac{c}{\sigma} \left(\frac{2-\alpha}{\alpha} \right) \frac{dT}{dn} \lambda \quad \dots\dots\dots (I.11)$$

APPENDIX II: THE SOLUTION FOR HEAT TRANSFER IN
THE INCOMPRESSIBLE BOUNDARY LAYER

The term 'incompressible' is of limited validity if applied to a boundary layer with heat transfer, since if there is a change in temperature within the layer there must also be a change in the density. However, if we apply the 'incompressible' condition $M = 0$ to the energy equation (3.4), it becomes

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \dots\dots\dots (II.1)$$

the dissipation of energy and the rate of doing work against the external pressure gradient being in this condition zero. If we now assume that the change in temperature is sufficiently small that we may linearise equation (II.1), it adopts the more usual form:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \left(\frac{k}{\rho c_p} \right) \frac{\partial^2 T}{\partial y^2} \dots\dots\dots (II.2)$$

where u and v are sensibly the velocity components within the layer if $T = T_a$, everywhere, and $(k/\rho c_p)$ may be treated as a constant. As the equation is linear we may superpose on the temperature distribution obtained for the boundary layer in the absence of a temperature jump, an additional term to account for the effects of the jump (i.e. of the difference between conditions for $T_s = T_w(x)$ and $T_s = T_\infty(x)$.)

One of the most general solutions of (II.2) has been obtained by Fage and Falkner⁵; they assumed that

$$\left. \begin{aligned} T_w = T_s = T_a + Bx^t \\ \text{and } u_0 = Cx^r \end{aligned} \right\} \dots\dots\dots (II.3)$$

and by an approximate treatment they showed that the heat transfer per unit area to the surface at a point x was of the form

$$Q = k \left(\frac{\partial T}{\partial y} \right)_s = - \left[\sigma P(r+1) \right]^{1/3} F \left(\frac{t}{r+1} \right) k (T_w - T_a) \left(\frac{\rho u_0}{\mu x} \right)^{1/2} \dots\dots (II.4)$$

where σ is the Prandtl number, F is a function shown in fig. 2, and

$$P \equiv P(r) = \frac{1}{2} (c_f R_x^{\frac{1}{2}}) \left(\frac{u_a}{u_0} \right)^{3/2}$$

/Now, from ...

Now, from (3.15), if $(T_w - T_a)$ is small it follows that

$$T_\zeta - T_w \propto \left(\frac{\partial T}{\partial y} \right)_s \propto Q$$

and since we may take $Q = Q^*$ here, within the accuracy of the first-order approximations, using (II.4)

$$T_\zeta - T_w \propto Q^* \left[x, 0, T_w(x) \right] \propto x^{t+\frac{r-1}{2}}$$

Suppose we wish to find the value of $Q^* \left[x, 0, T_\zeta(x) \right]$. This is the value, then, of the rate of heat transfer at the wall (in continuum flow) if the wall temperature is

$$\begin{aligned} T_\zeta &= T_a + (T_w - T_a) + (T_\zeta - T_w) \\ &= T_a + Bx^t + B'x^{t+\frac{r-1}{2}}, \text{ say.} \end{aligned}$$

Let us put

$$T - T_a = \Theta_1 + \Theta_2, \text{ say}$$

where $\Theta_1 = 0$ at $y = \infty$, $x > 0$, and $\Theta_1 = Bx^t$ at $y=0$, $x > 0$.
 $\Theta_2 = 0$ at $y = \infty$, $x > 0$, and $\Theta_2 = B'x^{t+\frac{r-1}{2}}$ at $y=0$, $x > 0$.

Since (II.2) is a linear differential equation in T , both Θ_1 and Θ_2 will obey the same equation, so that from (II.3) and (II.4)

$$k \left(\frac{\partial \Theta_1}{\partial y} \right)_{y=0} = - \left[\sigma P(r+1) \right]^{1/3} F \left(\frac{t}{r+1} \right) k Bx^t \left(\frac{\rho u_\delta}{\mu x} \right)^{1/2} = Q^* \left[x, 0, T_w(x) \right]$$

and similarly

$$k \left(\frac{\partial \Theta_2}{\partial y} \right)_{y=0} = - \left[\sigma P(r+1) \right]^{1/3} F \left[\frac{2t+r-1}{2(r+1)} \right] k B'x^{t+\frac{r-1}{2}} \left(\frac{\rho u_\delta}{\mu x} \right)^{1/2}$$

Hence

$$\begin{aligned} Q^* \left[x, 0, T_\zeta(x) \right] &= k \left(\frac{\partial T}{\partial y} \right)_s = k \left(\frac{\partial \Theta_1}{\partial y} \right)_{y=0} + k \left(\frac{\partial \Theta_2}{\partial y} \right)_{y=0} \\ &= Q^* \left[x, 0, T_w(x) \right] - \left[\sigma P(r+1) \right]^{1/3} F \left(\frac{r+t}{r+1} - \frac{1}{2} \right) k B'x^{t+\frac{r-1}{2}} \left(\frac{\rho u_\delta}{\mu x} \right)^{1/2} \end{aligned}$$

But $(T_\zeta - T_w) = B'x^{t+\frac{r-1}{2}}$, so that we can write

$$Q^* [x, 0, T_\zeta(x)] = Q^* [x, 0, T_w(x)] - [\sigma t(r+1)]^{1/3} F\left(\frac{r+t}{r+1} - \frac{1}{2}\right) k(T_\zeta - T_w) \left(\frac{\rho u_\delta}{\mu x}\right)^{1/2}$$

$$= \left[1 + \frac{F\left(\frac{r+t}{r+1} - \frac{1}{2}\right)}{F\left(\frac{t}{r+1}\right)} \left(\frac{T_\zeta - T_w}{T_w - T_a}\right) \right] Q^* [x, 0, T_w(x)]$$

..... (II.5)

In (4.8), with $Q \equiv G$, we have that

$$Q_y^* [x, 0, T_w(x)] = k \left(\frac{\partial^2 T}{\partial y^2} \right)_s = 0$$

and so, from (II.5),

$$Q_s = Q [x, 0, T_w(x)] = \left[1 + \frac{F\left(\frac{r+t}{r+1} - \frac{1}{2}\right)}{F\left(\frac{t}{r+1}\right)} \left(\frac{T_\zeta - T_w}{T_w - T_a}\right) \right] Q^* [x, 0, T_w(x)]$$

or if $Q_s = k_H^{1/2} \rho_a u_a^3$, using (3.15) it follows that

$$\Delta k_H = k_H - k_H^* = 2c \frac{F\left(\frac{r+t}{r+1} - \frac{1}{2}\right)}{F\left(\frac{t}{r+1}\right)} \left[\left(\frac{2\gamma}{\gamma+1}\right) \left(\frac{2-a}{a\sigma}\right) - \left(\frac{2-f}{f}\right) \right] \frac{\lambda Q_s^* k_H^*}{k(T_w - T_a)}$$

Thus, from (4.17), since $T_{th} = T_a$ if $M = 0$,

$$\frac{\Delta(Nu)}{(Nu^*)} = \frac{\Delta k_H}{k_H^*} = - \sqrt{\frac{\gamma\pi}{2}} \left[\frac{F\left(\frac{r+t}{r+1} - \frac{1}{2}\right)}{F\left(\frac{t}{r+1}\right)} \right] \left[\left(\frac{2\gamma}{\gamma+1}\right) \left(\frac{2-a}{a\sigma}\right) - \left(\frac{2-f}{f}\right) \right]$$

$$\times \left(\frac{p_a}{p_\delta}\right) \left(\frac{Nu^*}{R^2}\right) \sqrt{\frac{M^2}{R}} \quad \text{..... (II.6)}$$

where $Nu^* = -Q^* L / k(T_w - T_a) = -\left(\frac{\gamma-1}{2}\right) k_H^* \sigma R M^2 / \left(\frac{T_w - T_a}{T_a}\right)$, is the Nusselt number. It will be noticed from fig. 2 that since it is found that

$$\frac{2\gamma}{\gamma+1} \left(\frac{2-a}{a\sigma}\right) > \frac{2-f}{f}$$

in general (see para. 4.5), the sign of the correction to the heat transfer depends on the values of r and t .

APPENDIX III: AN EXTENSION TO THE COMPRESSIBLE
BOUNDARY LAYER WITH HEAT TRANSFER

As noted in the text, the effect of the temperature jump on the structure of the compressible boundary layer (i.e. that existing for $M \neq 0$) cannot be accurately described, unless several simplifying assumptions are made which restrict the application of the results.

A set of conditions which considerably simplify the analysis, has been suggested by Chapman and Rubesin⁶. If we are concerned with an isobaric flow in which

$$\frac{dp}{dx} = 0, \quad p_0 = p_a \quad \dots\dots\dots (III.1)$$

such as that over a flat plate, in a gas having constant specific heats, and so also constant Prandtl number; i.e.

$$\frac{dc_p}{dT} = \frac{dc_v}{dT} = \frac{d\sigma}{dT} = 0; \quad \dots\dots\dots (III.2)$$

and if the gas viscosity varies linearly with the temperature, i.e.

$$\omega = 1, \quad \dots\dots\dots (III.3)$$

so that $(\rho\mu)$ is a constant: then it may be shown that the shear stress, regarded as a function of the independent variables u and x , is independent of the temperature distribution. Thus the shear stress at the wall is uninfluenced by the temperature jump so that the correction due to the existence of a non-zero mean free path is precisely the same as that calculated in para. 4.2; or, by virtue of the condition (III.1), from (4.18),

$$\Delta c_f = 0 \quad \dots\dots\dots (III.4)$$

Chapman also shows that if the temperature of the wall is given by

$$T_w = T_{th}^* + T_a \sum_t B_t s^t \quad \dots\dots\dots (III.5)$$

where T_{th}^* is the thermometer temperature uncorrected for slip effects i.e.

$$T_{th}^* = T_a + r(\sigma) \frac{u_a^2}{2c_p}$$

where $r(\sigma)$ is the so-called 'recovery factor' which depends only on σ , and has the value 1 for unit Prandtl number; then the rate of heat transfer, uncorrected for slip effects is

$$Q^* [x, 0, T_w(x)] = -k_a T_a \sqrt{\frac{\rho_a u_a}{\mu_a x}} \sum_t F(\sigma, t) B_t s^t \dots\dots\dots (III.6)$$

The function $F(\sigma, t)$ has been evaluated exactly by Chapman for $\sigma = 0.72$: a comparison with (II.4) shows that, according to the approximations introduced by Fage and Falkner,

$$F(\sigma, t) = \left[\frac{\sigma}{2} c_f R_x^{\frac{1}{2}} \right]^{1/3} F(t), \text{ approximately}$$

$$\text{and since } c_f = 0.664 / R_x^{\frac{1}{2}} \dots\dots\dots (III.7)$$

it follows that

$$F(\sigma, t) \approx 0.693 \sigma^{1/3} F(t) \dots\dots\dots (III.8)$$

The two solutions should correspond since both cover the condition of heat transfer in the absence of a pressure gradient at low Mach number. In fact, the correspondence can be derived from inspection of the relevant differential equations used in refs. 5 and 6. A comparison of Chapman's values for $\sigma = 0.72$ and those of Fage and Falkner suggests an accuracy in the latter's treatment of some 2 per cent at $t = 0$; the error is zero for large values of t , and for $t = -\frac{1}{2}$. For our present purposes Fage and Falkner's approximation should therefore be adequate.

To estimate the effects of slipping and temperature jump we need the value of $(T_\zeta - T_w)$ which from (3.15) and (4.17) is

$$T_\zeta - T_w = \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha\sigma} \right) - \left(\frac{2-f}{f} \right) \right] \sqrt{\frac{\gamma\pi}{2}} \frac{M}{R} \frac{T_w}{T_a} \left(\frac{\partial T}{\partial y} \right)_s L$$

or since $k \propto \mu \propto T$, we find that $\frac{T_w}{T_a} = \frac{k_s}{k_a}$, and since $Q = k_s \left(\frac{\partial T}{\partial y} \right)_s$ it follows that

$$\frac{T_\zeta - T_w}{T_a} = \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha\sigma} \right) - \left(\frac{2-f}{f} \right) \right] \sqrt{\frac{\gamma\pi}{2}} \frac{M}{R} \frac{Q L}{k_a T_a} \dots (III.9)$$

To a sufficient approximation we may here put $Q = Q^*$, so that from (III.5), (III.6), and (III.9),

$$\begin{aligned} T_\zeta &= T_w - \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha\sigma} \right) - \left(\frac{2-f}{f} \right) \right] \sqrt{\frac{\gamma\pi}{2}} \frac{M}{R^{\frac{1}{2}}} \frac{T_a}{s^{\frac{1}{2}}} \sum_t F(\sigma, t) B_t s^t \\ &= T_{th}^* + T_a \sum_t \left\{ B_t s^t - \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha\sigma} \right) - \left(\frac{2-f}{f} \right) \right] \sqrt{\frac{\gamma\pi}{2}} \frac{M}{R^{\frac{1}{2}}} \left[F(\sigma, t) B_t \right] s^{t-\frac{1}{2}} \right\} \end{aligned}$$

By analogy with (III.5) and (III.6), it follows that

$$Q^* [x, 0, T_\zeta(x)] = -k_a T_a \sqrt{\frac{\rho_a u_a}{\mu_a x}} \sum_t \left\{ F(\sigma, t) B_t s^t - \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha \sigma} \right) - \left(\frac{2-f}{f} \right) \right] \sqrt{\frac{\gamma \pi}{2}} \frac{M}{R^{\frac{1}{2}}} F(\sigma, t^{-\frac{1}{2}}) \left[F(\sigma, t) B_t \right] s^{t-\frac{1}{2}} \right\}$$

In (4.8), if $G \equiv Q$, it follows from (3.4) that

$$Q_y^* [x, 0, T_w(x)] = - \frac{1}{\mu_s} P_s^{\#2}$$

so that

$$Q [x, 0, T_w(x)] = Q^* [x, 0, T_w(x)] - \zeta \frac{P_s^{\#2}}{\mu_s} + \frac{k_a T_a}{L} \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha \sigma} \right) - \left(\frac{2-f}{f} \right) \right] \times \sqrt{\frac{\gamma \pi}{2}} M \sum_t F(\sigma, t^{-\frac{1}{2}}) F(\sigma, t) B_t s^{t-1}$$

The value of ζ may be found from (3.14) and (4.17), so that from (III.3)

$$\frac{\zeta}{\mu_s} = \frac{L}{\mu_a} \sqrt{\frac{\gamma \pi}{2}} \left(\frac{2-f}{\gamma} \right) \frac{M}{R}$$

and hence in terms of the heat transfer coefficient,

$$\Delta k_H = k_H^* = - \sqrt{\frac{\gamma \pi}{2}} \left\{ \left(\frac{2-f}{2f} \right) M (c_f^*)^2 - \left(\frac{2}{\gamma-1} \right) \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha \sigma} \right) - \left(\frac{2-f}{f} \right) \right] \sum_t F(\sigma, t^{-\frac{1}{2}}) F(\sigma, t) \frac{B_t s^{t-1}}{\sigma M R} \right\} \dots \dots \dots (III.10)$$

This expression may be rewritten in a variety of forms. We note that, using (III.6)

$$\left. \begin{aligned} k_H^* &= \frac{Q_s^*}{\frac{1}{2} \rho_a u_a^3} = - \left(\frac{2}{\gamma-1} \right) \frac{1}{\sigma M^2 R^{\frac{1}{2}}} \sum_t F(\sigma, t) B_t s^{t-\frac{1}{2}} \\ \text{and} \\ Nu^* &= \frac{Q_s^* L}{k_a (T_{th}^* - T_w)} = + R^{\frac{1}{2}} \left[\sum_t F(\sigma, t) B_t s^{t-\frac{1}{2}} \right] \left[\sum_t B_t s^t \right] \end{aligned} \right\} (III.11)$$

and so in (III.10), from (III.8)

$$R^{\frac{1}{2}} \Delta k_H = - \sqrt{\frac{\gamma \pi M^2}{2R}} \left\{ \left(\frac{2-f}{2f} \right) (R^{\frac{1}{2}} c_f^*)^2 + \left[\left(\frac{2\gamma}{\gamma+1} \right) \left(\frac{2-\alpha}{\alpha \sigma} \right) - \left(\frac{2-f}{f} \right) \right] \Phi k_H^* Nu^* \right\} \dots \dots \dots (III.12)$$

/where ...

where
$$\bar{Q} = \frac{\left[\sum_t F(t-\frac{1}{2}) F(t) B_t s^{t-1} \right] \left[\sum_t B_t s^t \right]}{\left[\sum_t F(t) B_t s^{t-\frac{1}{2}} \right]^2}$$

The thermometer temperature (i.e. that reached by the surface when the heat transfer becomes zero) will be modified by slip effects. If we make $k_H = 0$, then $\Delta k_H = -k_H^*$ and so from (III.7), (III.11) and (III.12) to a first order of approximation, we have that

$$-\frac{2}{\gamma-1} \frac{1}{\alpha M^2} \sum_t F(\sigma, t) B_t s^{t-\frac{1}{2}} = 0.44 \sqrt{\frac{\gamma \pi}{2}} \frac{M}{R^{\frac{1}{2}}} \left(\frac{2-f}{2f} \right) \frac{1}{s} \dots (III.13)$$

Evidently, this equation is satisfied by a temperature distribution given by taking $t = -\frac{1}{2}$ but since $F(\sigma, -\frac{1}{2}) = 0$ it follows that the value of

$$\frac{T_{th} - T_{th}^*}{T_a} = B s^{-\frac{1}{2}} \dots \dots \dots (III.14)$$

is infinite to a first approximation. This is plainly incompatible with the assumption that the effects of slip are small, and it is pertinent to investigate the reasons why this incompatibility arises. It is in fact accounted for by the fact that the solution first outlined in ref. 5 which evaluates $F(\sigma, t)$ is, in fact, indeterminate for $t < 0$; the values implied in fig. 2 for $F(t)$, which is related approximately to $F(\sigma, t)$ by (III.8), are obtained by making the assumption that the air temperature is that of the free stream for all values of $y \neq 0$ in the plane of the plane leading edge ($s = 0$). This point is brought out in Appendix IV (in particular relation to the condition $t = -\frac{1}{2}$). If this condition is not satisfied then the rate of heat transfer connected with a given temperature distribution is unstipulated by the other boundary conditions imposed, and the temperature decays to its free-stream value far less rapidly away from the surface than does the velocity - that is, the thermal layer extends far beyond the velocity layer and the displacement thickness is infinite.

Such a condition may reasonably be assumed not to exist, implying as it does a heating of the flow outside the retarded boundary layer, and in the case of a given temperature distribution along the surface such an assumption leads to no anomaly. For although in the particular case that $t = -\frac{1}{2}$, the local rate of heat transfer may as a result be in general zero, the solution leads one to expect that a finite quantity of heat is transferred and that the transfer takes place on the surface

/at $x = 0 \dots$

at $x = 0$; the exact quantity depends on the magnitude of the variation in surface temperature. In other words corresponding to a given value of B in

$$T_w = T_{th}^* + B s^{-\frac{1}{2}}$$

there corresponds a certain total rate of heat transfer, the transfer taking place in the immediate vicinity of the nose. In fact, this form of temperature distribution evidently corresponds to a distribution of heat transfer which might be described by an impulse function, but not by a simple inverse proportion to x , such as is required to solve equation (III.13).

Evidently then to solve the inverse problem, that of the surface temperature distribution required to give a rate of heat transfer varying inversely to x , we must abandon the assumption that leads to an infinite surface temperature, and we conclude that the thermal layer must extend beyond the velocity layer.

To emphasise these remarks we may quote the following result, which is calculated from the data in Appendix IV: the temperature distribution at a distance y from the surface commensurate with the length $L\sqrt{s}$, is given in the condition of zero heat transfer by

$$T|_{k_{H=0}} - T_a \sim 0.22(\gamma-1) \sqrt{\frac{\gamma\pi}{2}} \left(\frac{2-f}{f}\right) \frac{M^3}{R} \left(\frac{L}{y}\right) T_a \dots (III.15)$$

whereas in a flow with no surface slip

$$T|_{k_{H=0}} - T_a \sim \text{const} \times \exp\left(-\frac{\sigma R y^2}{4 x L}\right) \times \left(\frac{x L}{R y^2}\right)$$

and $(u-u_a)$ decays in a similar rapid manner.

The physical significance of the condition expressed by (III.15) is perhaps small: the equilibrium condition of zero heat transfer may only theoretically be reached if the surface is a perfect reflector of heat and after the lapse of an infinite time. However the fact that to achieve this condition, an infinite transfer of heat must take place to the ambient air, - as we would infer from equation (III.15) - signifies that this equilibrium condition is reached much more slowly in the presence of slip. This conclusion follows as well from the known reduction of the rate of heat transfer, for a given surface temperature, due to slip effects.

For this reason the precise determination of the thermometer temperature in the presence of slip is of little

/importance. ...

importance. It would be given by a relation of the form of equation (III.14) where the value of $B^{-\frac{1}{2}}$ must be derived from a knowledge of the thermometer temperature at some point near the nose, where however the present method of analysis fails.

A description of the mathematical background to the remarks made in connection with this problem is given in Appendix IV.

$$(I.V) \dots \dots \dots \left\{ \begin{array}{l} u = v \text{ at } y = 0, \\ u = v = 0 \text{ at } y = 0. \end{array} \right.$$

$$(I.VI) \dots \dots \dots \left\{ \begin{array}{l} T = 1 \text{ at } y = 0, \\ T \text{ or } \frac{\partial T}{\partial y} \text{ known at } y = 0. \end{array} \right.$$

The relevant equations of the boundary layer are (2.1) to (2.4), and the physical conditions of the flow are described by equations (2.5) and (2.6). The development of the subsequent analysis follows closely that made in ref. 2.

Equation (2.1) is satisfied identically by the definition of a non-dimensional stream function ψ so that $\psi = 0$ corresponds to $y = 0$ and further that:

$$(2.V) \dots \dots \dots \left(\frac{\partial \psi}{\partial x} \right) = \frac{u}{U_\infty} \quad \left(\frac{\partial \psi}{\partial y} \right) = \frac{v}{U_\infty} \quad \left(\frac{\partial^2 \psi}{\partial y^2} \right) = \frac{1}{U_\infty} \frac{\partial u}{\partial y}$$

If we introduce a new set of independent variables (ξ, η) in place of (x, y) we have that

$$(2.VI) \dots \dots \dots \left\{ \begin{array}{l} \frac{\partial \psi}{\partial \xi} = \frac{u}{U_\infty} \quad \frac{\partial \psi}{\partial \eta} = \frac{v}{U_\infty} \\ \frac{\partial^2 \psi}{\partial \eta^2} = \frac{1}{U_\infty} \frac{\partial u}{\partial \eta} \end{array} \right.$$

Further we define the non-dimensional dependent variables θ, δ so that

$$(2.VII) \dots \dots \dots \theta = \frac{T - T_\infty}{T_0 - T_\infty} \quad \delta = \frac{\rho - \rho_\infty}{\rho_0 - \rho_\infty} \quad \rho_0 = \rho \text{ at } y = 0$$

Then, equations (2.1) and (2.4) become, using (2.VI) and (2.VII),

$$(2.VIII) \dots \dots \dots \left(\frac{\partial \psi}{\partial \xi} \right) = \frac{u}{U_\infty} \quad \left(\frac{\partial \psi}{\partial \eta} \right) = \frac{v}{U_\infty}$$

/APPENDIX IV ...

$$(2.VIII) \dots \dots \dots \frac{\partial \theta}{\partial \xi} = \frac{1}{U_\infty} \frac{\partial u}{\partial \xi} \quad \frac{\partial \theta}{\partial \eta} = \frac{1}{U_\infty} \frac{\partial v}{\partial \eta}$$

... first ...

APPENDIX IV: A SOLUTION OF THE BOUNDARY LAYER
EQUATIONS WITH ARBITRARY SURFACE TEMPERATURE

We shall here consider the problem of the solution of the boundary layer equations, uncorrected for slip, subject to the boundary conditions

$$\left. \begin{aligned} u &= u_a \quad \text{at } y = \infty, \\ u &= v = 0 \quad \text{at } y = 0; \end{aligned} \right\} \dots\dots\dots (IV.1)$$

$$\left. \begin{aligned} T &= T_a \quad \text{at } y = \infty, \\ \text{and } T \text{ or } k \frac{\partial T}{\partial y} &\text{ known at } y = 0. \end{aligned} \right\} \dots\dots\dots (IV.2)$$

The relevant equations of the boundary layer are (3.1) to (3.4), and the physical conditions of the flow are described by equations (4.27) and (4.28). The development of the consequent analysis follows closely that made in ref. 6.

Equation (3.1) is satisfied identically by the definition of a non-dimensional stream function ψ so that $\psi = 0$ corresponds to $y = 0$ and further that:

$$u = \frac{\sqrt{\mu_a u_a L \rho_a}}{\rho} \left(\frac{\partial \psi}{\partial y} \right), \quad v = - \frac{\sqrt{\mu_a u_a L \rho_a}}{\rho} \left(\frac{\partial \psi}{\partial x} \right) \dots\dots (IV.3)$$

If we introduce a new set of independent variables (ψ, x) in place of (x, y) we have that

$$\left. \begin{aligned} \frac{\partial}{\partial y} &= \frac{\rho u}{\sqrt{\rho_a \mu_a u_a L}} \frac{\partial}{\partial \psi} \\ \frac{\partial}{\partial x} \Big|_y &= \frac{\partial}{\partial x} \Big|_\psi - \frac{\rho v}{\sqrt{\rho_a \mu_a u_a L}} \frac{\partial}{\partial \psi} \end{aligned} \right\} \dots\dots\dots (IV.4)$$

Further we define the non-dimensional dependent variables s, ϕ' and θ , so that

$$s = x/L, \quad \phi' = 2(u/u_a), \quad \theta = T/T_a = \mu/\mu_a = \rho_a/\rho \dots\dots\dots (IV.5)$$

Then, equations (3.2) and (3.4) become, using (IV.4) and (IV.5),

$$2 \frac{\partial \phi'}{\partial s} = \frac{\partial}{\partial \psi} \left(\phi' \frac{\partial \phi'}{\partial \psi} \right) \dots\dots\dots (IV.6)$$

and

$$4 \frac{\partial \theta}{\partial s} = \frac{2}{\sigma} \frac{\partial}{\partial \psi} \left(\phi' \frac{\partial \theta}{\partial \psi} \right) + (\gamma-1) M^2 \phi' \frac{\partial \phi'}{\partial \psi} \dots\dots\dots (IV.7)$$

/The first ...

The first of these equations, the momentum equation (IV.6), is independent of the temperature distribution, and hence its solution is independent of Mach number. Its solution must therefore be the well-known Blasius solution: so let us define a new independent variable η such that:

$$\phi' = \frac{d\phi}{d\eta} \text{ where } \phi \equiv \phi(\eta) = \frac{\psi}{s^{\frac{1}{2}}} \dots\dots\dots (IV.8)$$

Thus ϕ' is a function only of η , and since from (IV.8)

$$\frac{\partial \eta}{\partial s} = - \frac{\phi}{2s\phi'}, \quad \frac{\partial \eta}{\partial \psi} = \frac{1}{s^{\frac{1}{2}}\phi'}$$

it follows in (IV.6) that

$$\phi''' + \phi\phi'' = 0 \dots\dots\dots (IV.9)$$

where the primes denote differentiations with respect to η .

Now, from (IV.3) and (IV.5)

$$R^{\frac{1}{2}} \frac{y}{L} = \int_0^{\psi} \left(\frac{\theta}{\phi'} \right) \Big|_s d\psi = s^{\frac{1}{2}} \int_0^{\eta} \theta \Big|_s d\eta \dots\dots\dots (IV.10)$$

so that y is infinite if ψ is infinite as well as η . From (IV.3) we also calculate that

$$\frac{v}{u_a} = \frac{1}{R^{\frac{1}{2}}} \left\{ \phi' s^{\frac{1}{2}} \int_0^{\eta} \left(\frac{\theta}{2s} + \frac{\partial \theta}{\partial s} \right) \Big|_s d\eta - \frac{\phi\theta}{2s^{\frac{1}{2}}} \right\} \dots\dots (IV.11)$$

Thus the boundary conditions designated by (IV.1) are that

$$\left. \begin{aligned} \phi' &= 2 \text{ at } \eta = \infty \\ \phi &= \phi' = 0 \text{ at } \eta = 0 \end{aligned} \right\} \dots\dots\dots (IV.12)$$

The integration of equation (IV.9) with the boundary conditions given by (IV.12) yields the familiar Blasius solution.

In terms of the independent variables η and s , the equation (IV.7) may be written

$$\frac{\partial^2 \theta}{\partial \eta^2} + \phi \frac{\partial \theta}{\partial \eta} - 2\phi' s \frac{\partial \theta}{\partial s} = - \frac{\sigma}{4} (\gamma-1) M^2 (\phi'')^2 \dots\dots\dots (IV.13)$$

A particular integral for θ is given by

$$\left. \begin{aligned} \theta &= 1 + \frac{\gamma-1}{2} M^2 r(\sigma, \eta) \\ \text{where } r(\sigma, \eta) &= \frac{\sigma}{2} \int_{\eta}^{\infty} [\phi''(\alpha)]^{\sigma} \int_0^{\alpha} [\phi''(\beta)]^{2-\sigma} d\beta d\alpha \end{aligned} \right\} \dots (IV.14)$$

This solution satisfies the conditions that

$$\left. \begin{aligned} \theta &= 1 \text{ at } \eta = \infty \quad \text{i.e. } T = T_a \text{ at } y = \infty \\ \theta &= 1 + \frac{\gamma-1}{2} M^2 r(\sigma, 0) \text{ at } \eta=0 \quad \text{i.e. } T = T_{th}^* \text{ at } y=0 \\ \theta_{\eta} &= 0 \text{ at } \eta=0 \quad \text{i.e. } \frac{\partial T}{\partial y} = 0 \text{ at } y=0 \end{aligned} \right\} (IV.15)$$

and is evidently the temperature distribution existing if the heat transfer to the surface is zero. We now write

$$\theta = 1 + \frac{\gamma-1}{2} M^2 r(\sigma, \eta) + \Theta(s, \eta) \dots (IV.16)$$

and then in (4.18)

$$\frac{\partial^2 \Theta}{\partial \eta^2} + \partial \phi \frac{\partial \Theta}{\partial \eta} - 2\sigma \phi' s \frac{\partial \Theta}{\partial s} = 0 \dots (IV.17)$$

From (IV.15) and (IV.16) it follows that to satisfy the boundary conditions given by (IV.2) we must have

$$\Theta = 0 \text{ for } \eta = \infty \dots (IV.18)$$

The boundary condition at the surface must be interpreted either as

$$\Theta(s, 0) = \frac{T_w - T_{th}}{T_a}, \quad \text{or } \Theta_{\eta}(s, 0) = \frac{1}{k_a T_a} \frac{L s^{\frac{1}{2}}}{R^{\frac{1}{2}}} \left(k \frac{\partial T}{\partial y} \right)_s \dots (IV.19)$$

We shall consider two problems, given by selecting either of the two boundary conditions:

$$\frac{T_w - T_{th}}{T_a} = B s^{-\frac{1}{2}} \dots (i)$$

$$\left(k \frac{\partial T}{\partial y} \right)_s = \frac{Q}{s} \dots (ii)$$

If we solve (IV.17) by separation of variables, in either problem we must put

$$\Theta = s^{-\frac{1}{2}} N(\eta) \dots (IV.20)$$

/and then, ...

and then, in (IV.17),

$$N'' + \sigma \phi N' + \sigma \phi' N = 0$$

or upon integration, since from (IV.15), $\exp \left(\int \phi \, d\eta \right) = 1/\phi''$

$$N(\eta) = N'(0) \left[\phi''(\eta) \right]^\sigma \int_0^\eta \left[\phi''(\alpha) \right]^{-\sigma} d\alpha + N(0) \left[\frac{\phi''(\eta)}{\phi''(0)} \right]^\sigma \dots (IV.21)$$

the condition that $N(\infty) = 0$ as required by (IV.18), being satisfied for any value of $N'(0)$ or $N(0)$. It follows that the boundary conditions given by (IV.18) and (IV.19) are insufficient.

The behaviour for large η must therefore be investigated so that the uniqueness of the solution may be established. From (IV.9) and (IV.12)

$$\phi \sim 2\eta \quad \text{and} \quad \phi'' \sim A \exp(-\eta^2) \quad \text{for} \quad \eta \rightarrow \infty. \dots (IV.22)$$

and from (IV.10)

$$\eta \sim \frac{R^{\frac{1}{2}} y}{s^{\frac{1}{2}} L} \dots (IV.23)$$

From (IV.21), using (IV.23) it follows that, for $\eta \rightarrow \infty$

$$N(\eta) \sim \frac{N'(0)}{2\sigma\eta} + \frac{N(0)}{\left[\phi''(0) \right]^\sigma} e^{-\sigma\eta^2} \dots (IV.24)$$

since from (IV.14) as $\eta \rightarrow \infty$.

$$r(\sigma, \eta) \sim \frac{A}{4} \int_0^\infty \left[\phi''(\beta) \right]^{2-\sigma} d\beta \frac{e^{-\sigma\eta^2}}{\eta} = \frac{C e^{-\sigma\eta^2}}{\eta}, \text{ say, } \dots (IV.25)$$

it follows that from (IV.25), (IV.20) and (IV.24) in (IV.16) we have that

$$\theta - 1 \sim \frac{N'(0)}{2\sigma s^{\frac{1}{2}} \eta} + \left[\frac{N(0)}{\left[\phi''(0) \right]^\sigma s^{\frac{1}{2}}} + \frac{\gamma-1}{2} \frac{M^2 C}{\eta} \right] e^{-\sigma\eta^2} \quad (IV.26)$$

Thus, for $y \neq 0$, and $s \rightarrow 0$, since $\eta \rightarrow \infty$, from (IV.23) in (IV.26)

$$(\theta-1) \sim \frac{N'(0)L}{2\sigma R^{\frac{1}{2}} y} \dots (IV.27)$$

It therefore appears that to specify the solution completely we need one further condition

- (i) if the surface temperature distribution is known (problem (i)) we must specify the gas temperature for some finite value of $y \neq 0$ at $s = 0$.

(ii) ...

- (ii) if the surface heat flux is known (problem (ii)) we must specify the surface temperature for some value of $s \neq 0$ at $y = 0$. (This follows from equation IV.21)

In the first problem, we introduce the condition that $T = T_a$ for $s = 0$ and $y \neq 0$: thereby inferring that there can be no transfer of heat to the air outside the boundary layer. As a consequence, $N'(0) = 0$ from (IV.27) so that from (IV.19) and (IV.20) it will be seen that in this condition, i.e. with

$$\frac{T_w - T_{th}}{T_a} = Bs^{-\frac{1}{2}}$$

there is no transfer of heat to the body except at $s = 0$.

On the other hand, in the second problem - i.e. with

$$\left(k \frac{\partial T}{\partial y} \right)_s = \frac{Q}{s}$$

- the value of $N'(0)$ is essentially non-zero, so that there is an escape of heat into the air outside the boundary layer. At the same time there is an arbitrary distribution of the wall temperature, as described in the first problem above, and its value must be stipulated at one point away from the nose (i.e. for some $s \neq 0$).

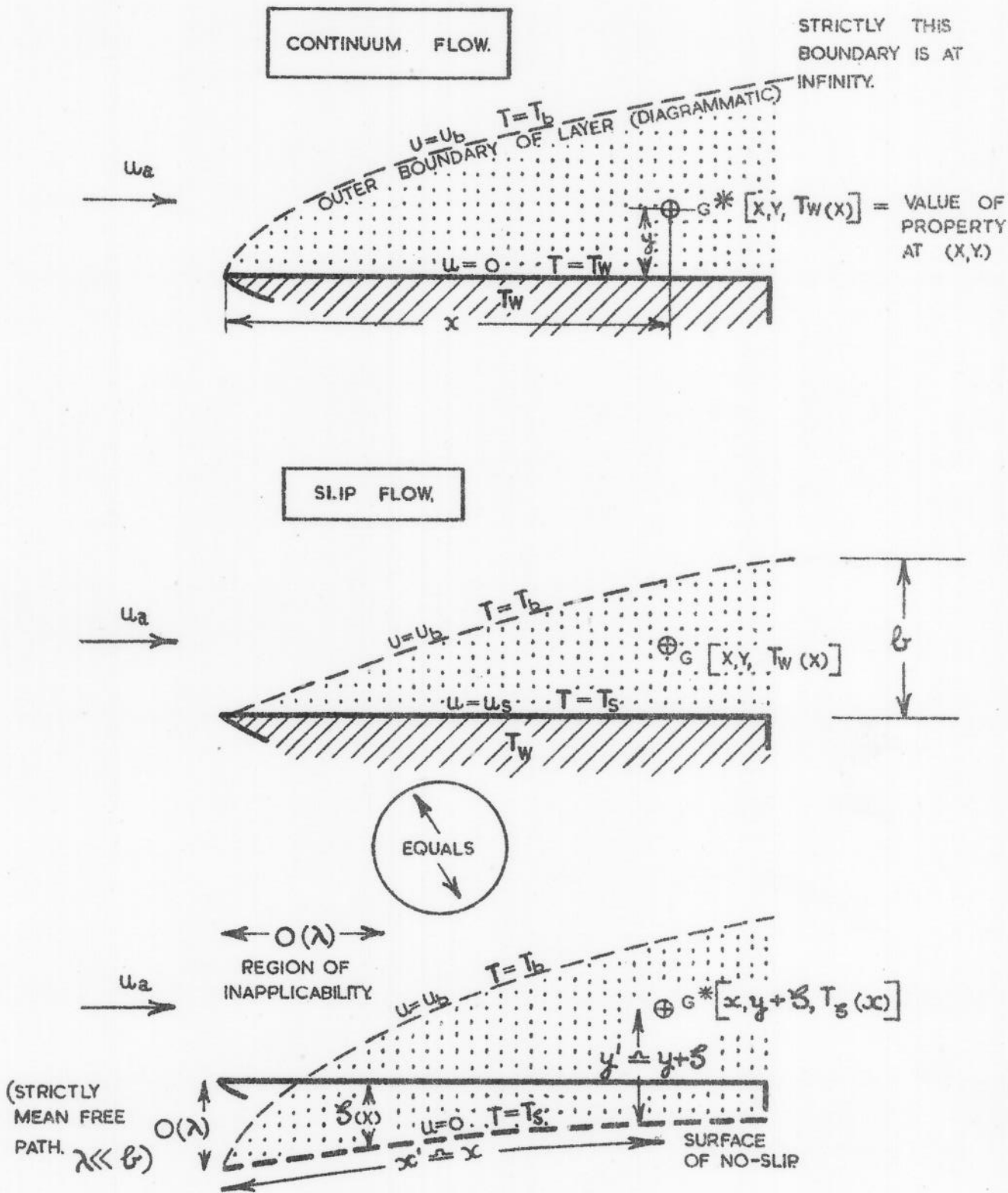


FIG. 1. DIAGRAMMATIC REPRESENTATION OF METHOD OF SOLUTION.

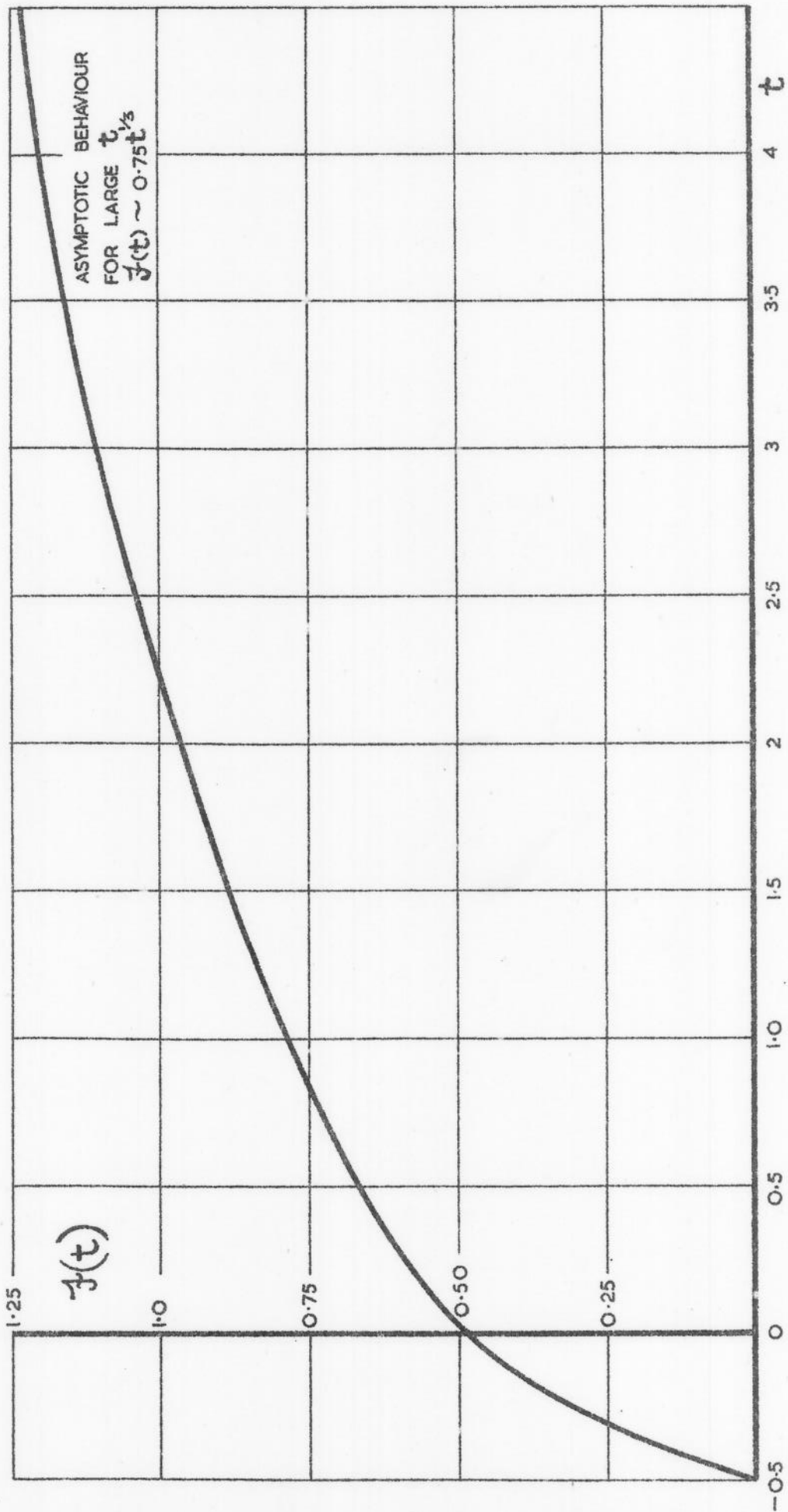


FIG. 2. VALUES OF FUNCTION $f(t)$ IN EQUATION (4.27) (TAKEN FROM REF 5.)